PHASE SEPARATION IN MULTIPLY PERIODIC MATERIALS WITH FINE MICROSTRUCTURES

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Abstract. We study a Cahn-Hilliard model for phase separation in composite materials with multiple periodic microstructures. These are modeled by considering a highly oscillating potential. The focus of this paper is in the case where the scales of the microstructures are smaller than that of phase separation. We provide a compactness result and prove that the Γ-limit of the energy is a multiple of the perimeter. In particular, using the recently introduced unfolding operator for multiple scales, we show that the taking the limit of all of the scales together is equivalent to taking one limit at the time, starting from the smaller scale and keeping the larger fixed.

1. Introduction

Composite materials are everywhere in natural (bones, blood, ice, wood) and in synthetic products (foam, colloids, concrete, elastomers). Therefore, there is a huge interest in understanding how to obtain an effective description of chemical, physical, and mechanical properties that are essential for applications, such as conductivity, stiffness, permeability. The complex interactions between the the different components result in a dependence of these effective properties on nontrivial details of the microstructure. In this manuscript, we focus on understanding the distribution of phases at stable equilibrium in a composite material with two periodic microstructures (both larger than the molecular scale) at different scales.

For a single material under isothermal conditions, the classical model used to describe stable configurations of phases in a liquid-liquid separation is the celebrated van der Waals–Cahn–Hilliard (also known as Modica–Mortola) functional defined as follows:

$$F_\varepsilon^{(0)}(u) := \int_\Omega \left[W(u) + \varepsilon^2 |\nabla u|^2\right] \mathrm{d}x \tag{1}$$

for $u \in W^{1,2}(\Omega;\mathbb{R}^M)$. Here, $\Omega \subset \mathbb{R}^N$ is an open bounded set, $\varepsilon > 0$ is a small parameter, and the continuous function $W : \mathbb{R}^M \to [0,+\infty)$ is the material dependent free energy density with suitable growth at infinity and vanishing at two points (the wells) $a, b \in \mathbb{R}^M$. These latter correspond to the stable phases. The main goal of the analysis is to understand the asymptotic behavior, as ε vanishes, of miminizers of F_ε under a mass constraint of the form

$$\fint_\Omega u \,\mathrm{d}x = ma + (1-m)b, \qquad m \in (0,1).$$

Using the expansion by Γ-convergence (see [4, 20]), it is has been proved (see [11, 35, 34, 40, 39, 26]) that

$$F_\varepsilon^{(0)} \approx F_\infty^{(0)} + \varepsilon F_\infty^{(1)}, \tag{2}$$

where

$$F_\infty^{(0)}(u) := \int_\Omega W(u)\,\mathrm{d}x,$$

and

$$F_\infty^{(1)}(u) := \sigma |Du|(\Omega),$$

for $u \in BV(\Omega;\{a,b\})$, namely functions of bounded variations taking values in the set $\{a,b\}$. Here,

$$\sigma := \inf\left\{\int_{-1}^{1} 2\sqrt{W(\gamma(t))}\,|\gamma'(t)|\;\mathrm{d}t : \gamma \in \mathrm{Lip}([-1,1];\mathbb{R}^M), \gamma(-1)=a, \gamma(1)=b\right\}. \tag{3}$$

This latter functional is the Γ-limit in a suitable L^p topology, depending on the growth of F at infinity, of

$$F^{(1)}_\varepsilon := \frac{F^{(0)}_\varepsilon - \min F^{(0)}_\infty}{\varepsilon} = \frac{F^{(0)}_\varepsilon}{\varepsilon}.$$

The expansion in (2) reads as follows: minimizers of $F^{(0)}_\varepsilon$ converges to minimizers of $F^{(1)}_\infty$, which are also minimizers of $F^{(0)}_\infty$, at an 'energy-rate' ε. Moreover, it turns out that the typical minimizer u_ε of $F^{(0)}_\varepsilon$ is a function with the following structure: take a set $E \subset \Omega$ such that the function $u := a\mathbb{1}_E + b\mathbb{1}_{\Omega\setminus E}$ minimizes the functional $F^{(1)}_\infty$ under the mass constraint $|E| = m|\Omega|$. Consider an ε-tubular neighborhood $(\partial E)_\varepsilon$ of the boundary of E. Then, $u_\varepsilon \in H^1(\Omega;\mathbb{R}^M)$ is equal to a in $E\setminus(\partial E)_\varepsilon$, to b in $\Omega\setminus(E\cup(\partial E)_\varepsilon)$, and has an optimal transition between those two values in $(\partial E)_\varepsilon$ that resembles the optimal profile that solves the minimization problem defining σ in (3). In particular, the transition region has size of order ε. Therefore, the expansion (2) yields an approximation both of minimizers and of their energy.

Several extensions of the model (1) have been studied over the years. Here, we limit ourselves to recall those considered by Baldo in [6] for the case of multiple wells, by Barroso and Fonseca in [7] for general singular perturbations, and by Owen and Sternberg in [38] and by Fonseca and Popovici in [25] for the case of fully coupled integrands. For a more complete review of the results on the topic, we refer the reader to the Introduction of [18].

In the case the material has *macroscopic* heterogeneities, for instance it is not in an isothermal case, the functional (1) has to be modified by taking into consideration the different response of the material at any given point to a specific phase. Namely, we consider the functional

$$F^{(0)}_\varepsilon(u) := \int_\Omega \left[W(x,u) + \varepsilon^2|\nabla u|^2\right]\mathrm{d}x, \tag{4}$$

for $u \in W^{1,2}(\Omega;\mathbb{R}^M)$, where $W : \Omega\times\mathbb{R}^M \to [0,\infty)$ is a continuous function with a suitable growth at infinity and close to its wells such that $W(x,p) = 0$ if and only if $p \in \{a(x), b(x)\}$, where $a,b : \Omega \to \mathbb{R}^M$ are Lipschitz functions. Also in such a case, an expansion of the form (2) is possible (see [8] for the scalar case, [18] for the vectorial case, and also [15] for a weaker sets of assumptions on the behavior of the potential W close to the wells), where now

$$F^{(1)}_\infty(u) := \int_{J_u} \sigma(x)\;\mathrm{d}\mathcal{H}^{N-1}(x),$$

for $u \in BV(\Omega;\{a,b\})$. Here, $BV(\Omega;\{a,b\})$ is the space of functions $u : \Omega \to \mathbb{R}^M$ of bounded variations such that $u(x) \in \{a(x), b(x)\}$ for a.e. $x \in \Omega$, J_u denotes the jump set of the function u, and

$$\sigma(x) := \inf\left\{\int_{-1}^{1} 2\sqrt{W(x,\gamma(t))}\,|\gamma'(t)|\;\mathrm{d}t : \gamma \in \mathrm{Lip}([-1,1];\mathbb{R}^M), \gamma(-1)=a(x), \gamma(1)=b(x)\right\}. \tag{5}$$

Namely, the function σ is the analogous of (3) when we 'freeze' the point $x \in J_u$.

We now enter into the realm of the modeling of phase separation in composite materials. We consider the case where the material has a periodic microstructure with scale $\delta > 0$ (see Figure 1 to the left). The natural modification of the functional (1) yields

$$F^{(0)}_{\varepsilon,\delta}(u) := \int_\Omega \left[W\left(\frac{x}{\delta}, u\right) + \varepsilon^2|\nabla u|^2\right]\mathrm{d}x. \tag{6}$$

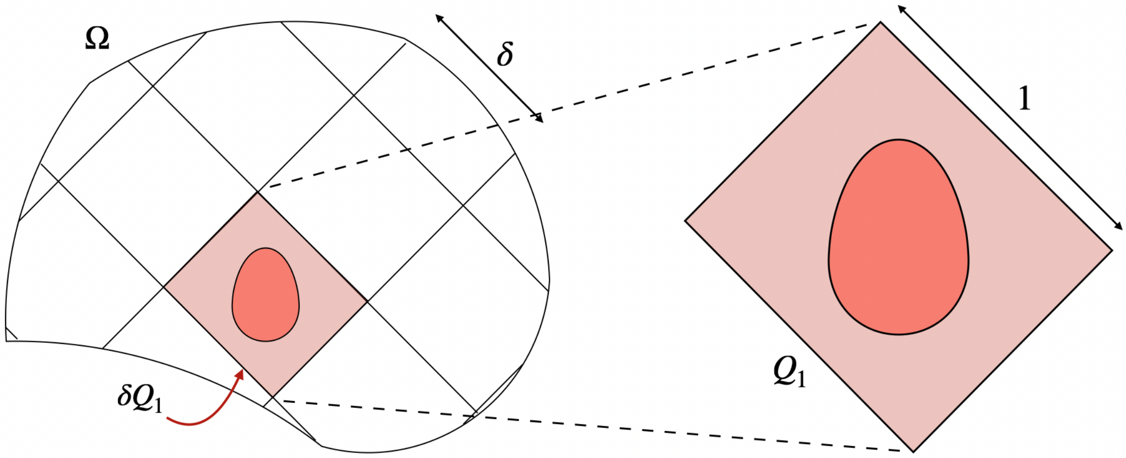

FIGURE 1. Left: A composite material with a periodic microstructure. Right: A microstructure with materials inclusions. Different colors correspond to different materials.

To model the periodicity models the periodic structure inside the periodicity cell Q_1 we require the potential $W : \Omega \times \mathbb{R}^M \to [0, \infty)$ to be a Carathéodory function that is Q_1-periodic in the first variable, with suitable growth at infinity, and with wells at $a, b \in \mathbb{R}^M$. In particular, the low regularity in the first variable is necessary in order to consider the case of a microstructure with materials inclusions $E \subset Q_1$ (see Figure 1 to the right). In such a case, the potential W might jump from one material to another.

Obtaining an expansion of the form (2) is now more challenging, due to the presence of two parameters in the problem. Indeed, there is a competition between the process of homogenization of the periodic structure and that of transition between the stable phases. The former happening at a scale δ, while the latter at an (expected) scale ε. Therefore, the problem naturally gives three regimes:

$$\delta \ll \epsilon \qquad\qquad \delta \approx \epsilon \qquad\qquad \epsilon \ll \delta.$$

The zeroth order in the expansion by Γ-convergence yields a functional of the form

$$F^{(0)}_\infty(u) := \int_\Omega W_r(u)\,\mathrm{d}x,$$

where the bulk energy density $W_r : \mathbb{R}^M \to [0, \infty)$ depends on which of the three above regimes we are in. Nevertheless, it always hold that

$$\min\left\{ F^{(0)}_\infty(u) : \fint_\Omega u\,\mathrm{d}x = ma + (1-m)b \right\} = 0.$$

The interesting term is $F^{(1)}_\infty$.

In the first regime, the homogenization process happens at a smaller scale then the phase separation one. Therefore, we expect to first the combined limit to be equivalent to first sending $\delta \to 0$ while keeping ε fixed, and then to send $\varepsilon \to 0$, namely first homogenize and then do phase separation. This was confirmed in [16], where it was proved that

$$F^{(1)}_\infty(u) := \sigma^{\mathrm{h}} |Du|(\Omega).$$

for $u \in BV(\Omega; \{a, b\})$, where

$$\sigma^{\mathrm{h}} := \inf\left\{ \int_{-1}^{1} 2\sqrt{W^{\mathrm{h}}(\gamma(t))}\,|\gamma'(t)|\,\mathrm{d}t : \gamma \in \mathrm{Lip}([-1,1];\mathbb{R}^M), \gamma(-1) = a, \gamma(1) = b \right\}, \tag{7}$$

and
$$W^{\mathrm{h}}(u) \coloneqq \fint_{Q_1} W(y,u)\,\mathrm{d}y$$
is the averaged potential, which can be seen as the homogenization of W with respect to the strong L^1 convergence. Note that the surface energy $F^{(1)}_\infty$ is isotropic.

In the second regime, namely when $\delta \approx \varepsilon$, the two physical processes interact at the same scale, and therefore a complex formula for the limiting energy
$$F^{(1)}_\infty(u) \coloneqq \int_{J_u} \sigma(\nu_u)\,\mathrm{d}\mathcal{H}^{N-1}, \tag{8}$$
for $u \in BV(\Omega;\{a,b\})$ is expected. Here, $\nu_u \in \mathbb{S}^{N-1}$ denotes the measure theoretical exterior unit normal to $\{u=a\}$. The precise formula for σ, in the case $\delta = L\varepsilon$, was obtained in [17], and reads as
$$\sigma_L(\nu) \coloneqq \lim_{T\to\infty} \frac{1}{T^{N-1}} \inf\left\{ \int_{TQ_\nu} \left[\frac{1}{L}W(x,u) + L|\nabla u|^2\right] \mathrm{d}z : u \in \mathcal{A}_T \right\}, \tag{9}$$
and
$$\mathcal{A}_T \coloneqq \left\{ u \in H^1(TQ_\nu;\mathbb{R}^M),\ u = \rho * u_\nu \text{ on } \partial(TQ_\nu) \right\},$$
where Q_ν is a cube with two faces orthogonal to ν, ρ is a standard mollifier, and $u_\nu(x)$ equals a if $x\cdot\nu > 0$, and b otherwise. It is worth noticing that $L \in (0,\infty)$, and that the limiting surface energy is anisotropic. The reason is that the direction ν of jump of the function u might be not aligned with the directions of periodicity of the potential W. This mismatch is at the origin of the anisotropic nature of the limiting energy. Note that, contrary to the previous regime, the optimal transition profile is no more one dimensional.

The third regime $\varepsilon \ll \delta$ sees the phase separation process happening at a smaller scale than homogenization. Thus, we heuristically expect to first send $\varepsilon \to 0$ while keeping δ fixed, and then send this latter to zero. In this case, with the first limit we go from a bulk energy to a surface energy of the form
$$E \mapsto \int_{\partial^* E} \sigma\left(\frac{x}{\delta}\right) \mathrm{d}\mathcal{H}^{N-1}(x), \tag{10}$$
for sets $E \subset \mathbb{R}^N$ of finite perimeter, and $\sigma : \Omega \to [0,\infty)$ is defined in (5). Note that, with this first limit, we pass from a *bulk* energy to a *surface* energy, that we now need to homogenize. This is done by using the theory of plane-like minimizers developed by Caffarelli and de la Llave in [10], and used by Chambolle and Thouroude in [12] in the context of homogenization. This gives, for each $\nu \in \mathbb{S}^{N-1}$, the existence of a set of locally finite perimeter $E_\nu \subset \mathbb{R}^N$ such that the homogenization of the energy in (11) is given by
$$E \mapsto \int_{\partial^* E} \sigma^{\mathrm{h}}_{surf}(\nu_E(x))\,\mathrm{d}\mathcal{H}^{N-1}(x), \tag{11}$$
where
$$\sigma^{\mathrm{h}}_{surf}(\nu) \coloneqq \lim_{T\to\infty} \frac{1}{T^{N-1}} \int_{TQ_\nu \cap \partial^* E_\nu} \sigma(y)\,\mathrm{d}\mathcal{H}^{N-1}(y). \tag{12}$$
The proof that the functional in (11) equals $F^{(1)}_\infty$ is provided in [16].

Note that the limiting surfaces densities in (7), (9), and (12) are all surface densities of a constant quantity, a bulk integral, and a surface one, respectively.

Finally, we remark that, in all of the cases, the limiting energy does not depend on the spatial variable. This is an advantage for both the theoretical and the numerical point of view. Indeed, for the former it gives a geometric model approximating a complex physical phenomenon, other than theoretical tools to investigate regularity properties of minimal interfaces. For the latter, numerical simulations for $F^{(1)}_\infty$ are extremely expensive when $\delta \ll 1$, since the size of the discretization grid has to be smaller than δ. This is in analogy with what happens in the classical theory of homogenization. Therefore, it is more convenient to perform numerical simulations for the functional $F^{(1)}_\infty$ in (8) than for (6). Of course, one has to compute the limiting surface energy densities (7), (9), and (12), and the last two are not that easy.

Extensions of the functional (6) in the context of phase separation in composite materials with one scale of microstructures have been considered by several researchers. In particular, the case where wells are also dependent on the spatial variable has been investigated by the first author, Fonseca and Ganedi in [15] in the regime $\varepsilon \ll \delta$. Moreover, the case where oscillations are in the singular term have been considered by Ansini, Braides, and Chiadò Piat in [3] (see also [2]), while the effect of an highly oscillating forcing term has been investigated by Dirr, Lucia, and Novaga in [21] and [22]. Finally, the literature for stochastic setting features recent contributions by Marziani (see [33]), by Bach, Marziani and Zeppieri (see [5]), by Morfe (see [36]), by Morfe and Wagner (see [37]) and by Donnarumma (see [23]).

The question that is at the basis of the project, of which this paper is the first step, is the following: what happens when the material has multiple microstructures at different scales? In particular, is it true that the 'principle of multiscale physics' obtained above for one scale and according to which we compute 'one limit at the time' holds true also for multiple scales?

The prototype of a composite material with multiple microstructures is that of a periodic structure with materials inclusions having a periodic (smaller) microstructure as well (see Figure 2 on the left). Namely, we have in mind potentials $W : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^M \to [0,\infty)$ of the form

$$W(y_1, y_2, z) := \mathbb{1}_{I_1}(y_1)[\mathbb{1}_{I_2}(y_2)W_1(u) + \mathbb{1}_{Q_2\setminus I_2}(y_2)W_2(u)] + \mathbb{1}_{Q_1\setminus I_1}(y_1)W_3(u),$$

that are Q_1 periodic in the first variable, and Q_2 periodic in the second, where Q_1 and Q_2 are the periodicity cells of the larger and the smaller microstructures. Note that there is no relation between Q_1 and Q_2. Therefore, we are naturally led to consider the functional

$$F^{(0)}_{\varepsilon,\delta,\eta}(u) := \int_\Omega \left[W\left(\frac{x}{\delta}, \frac{x}{\eta}, u\right) + \varepsilon^2 |\nabla u|^2 \right] \mathrm{d}x. \tag{13}$$

Having the above example in mind, we always assume a separation of scales

$$\eta \ll \delta.$$

A complete study of the Γ-convergence of this functional requires five distinct regimes:

(1) $\eta \ll \delta \ll \varepsilon$;

(2) $\eta \ll (\varepsilon \approx \delta)$;

(3) $\eta \ll \varepsilon \ll \delta$;

(4) $(\eta \approx \varepsilon) \ll \delta$;

(5) $\varepsilon \ll \eta \ll \delta$.

Note that the case of more than two microscales reduces to one of the cases above, except for the (family of) case(s) where $\lambda \ll (\varepsilon \approx \eta) \ll \delta$, being λ another scale. Nevertheless, this situation can be treated by combining what happens in the second and the fourth regimes above.

In this paper we focus on the regime

$$\eta \ll \delta \ll \varepsilon.$$

For this case, we expect to obtain the first order Γ-limit by first sending η to zero, then δ, and finally ϵ. We confirm this claim by a careful analysis that allows us to make rigorous the above procedure of taking 'one limit at the time'. The main novelty of the paper is in developments of techniques robust enough to be applied to multiple scales. In particular, in our main theorem (see Theorem 9) we show that

$$F^{(1)}_\infty(u) := \sigma^{\mathrm{h}} |Du|(\Omega), \tag{14}$$

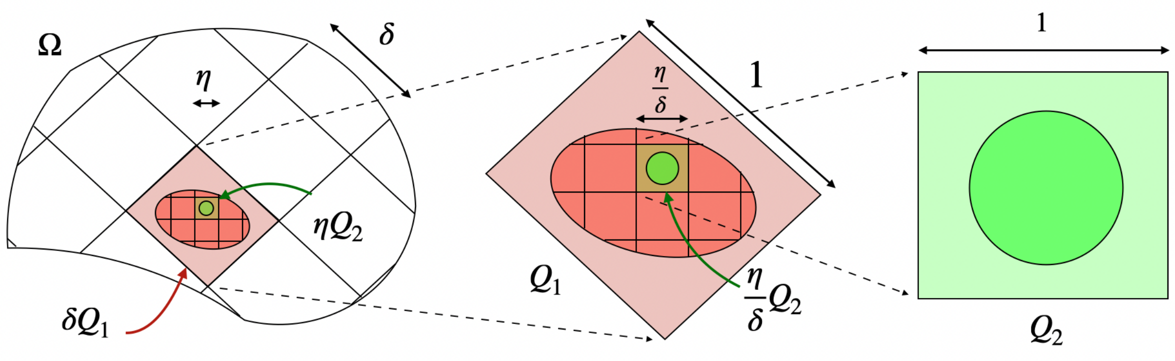

FIGURE 2. Left: A composite material with a two nested periodic microstructures. Center: A microstructure with materials inclusions and a nested microstructure. Right: A microstructure with periodic inclusions. Different colors correspond to different materials.

for functions $u \in BV(\Omega; \{a, b\})$, where

$$\sigma^{\mathrm{h}} := \inf\left\{ \int_{-1}^{1} 2\sqrt{W^{\mathrm{h}}(\gamma(t))}\, |\gamma'(t)| \,\mathrm{d}t : \gamma \in \mathrm{Lip}([-1,1]; \mathbb{R}^M), \gamma(-1) = a, \gamma(1) = b \right\},$$

and

$$W^{\mathrm{h}}(z) := \fint_{Q_1} \fint_{Q_2} W(y_1, y_2, z) \,\mathrm{d}y_1 \,\mathrm{d}y_2,$$

We also provide a compactness result (see Theorem 6) that allows us to use standard results of Γ-converge to prove the convergence of minimizers and minima. Finally, we consider the case where the functional (13) is finite only on configurations satisfying a mass constraint, and we prove that this passes to the limit; namely, that the first order Γ-limit is finite only for configurations satisfying the mass constraint and, in this case, given by the functional (14).

The other regimes are the focus of a series of forthcoming papers.

2. Assumptions and main results

Let G_δ, G_η be two subgroups of $\mathbb{R}^N$ with rank N, corresponding to the microscales δ and η. Let Q_1 be the periodicity cell with respect to G_δ, and let Q_2 be the periodicity cell with respect to G_η. We assume them to be bounded, with Lipschitz boundary, and containing the origin. Points in Q_1, Q_2 will be denoted by y_1, y_2, respectively.

Let $W\colon \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ be a function satisfying the following hypotheses:

(H1) W is a Carathéodory function, periodic in the spatial variables, that is:
- $z \mapsto W(y_1, y_2, z)$ is continuous for $\mathcal{L}^N$-a.e. $y_1 \in Q_1$, $y_2 \in Q_2$;
- $y_1 \mapsto W(y_1, y_2, z)$ is measurable and G_1-periodic for all $z \in \mathbb{R}^M$ and for $\mathcal{L}^N$-a.e. $y_2 \in Q_2$, namely $W(y_1 + \xi_1, y_2, z) = W(y_1, y_2, z)$ for all $\xi_1 \in G_1$;
- $y_2 \mapsto W(y_1, y_2, z)$ is measurable and G_2-periodic for all $z \in \mathbb{R}^M$ and for $\mathcal{L}^N$-a.e. $y_1 \in Q_1$, namely $W(y_1, y_2 + \xi_2, z) = W(y_1, y_2, z)$ for all $\xi_2 \in G_2$.

(H2) There exist $a, b \in \mathbb{R}^M$ such that

$$W(y_1, y_2, z) = 0 \iff z \in \{a, b\}.$$

(H3) There exists $R > 0$ such that for $\mathcal{L}^N$-a.e. $y_1 \in Q_1$, $y_2 \in Q_2$, if $|z| \geq R$ then it holds:

$$W_1(z) \geq \frac{1}{R}|z|.$$

(H4) For every $S>0$, there exists a constant $C_S>0$ depending only on S such that

$$\operatorname{ess\,sup}_{y_1\in Q_1, y_2\in Q_2, |z|\leq S} W(y_1,y_2,z)\leq C_S.$$

(H5) There exists $W_1:\mathbb{R}^M\to\mathbb{R}$ such that

$$0\leq W_1(z)\leq W(y_1,y_2,z)\qquad \forall y_1\in Q_1, \forall y_2\in Q_2.$$

Remark 1. We note that Assumption (H5) is only needed for the compactness. In [17, Theorem 1.6], the compactness was obtained without the need of a spatially uniform lower bound on W. Despite it is possible to use a similar strategy as employed in [17, Theorem 1.6] to get compactness without the need of a lower bound, we prefer to add this assumption in order to focus on the strategy to get the Γ-limit.

Remark 2. The hypothesis (H4) seems quite natural to us, as it implies that the potential energy does not blow up in a finite space. It will be used in the proofs to bound some integral terms containing W. A similar assumption appears in the work [6] by Baldo (see formula (1.2) in that paper).

Remark 3. Hypothesis (H3) is needed in order to get compactness. This is done by using the classical strategy developed in [26] by Fonseca and Tartar (see also [30]). In the case where a mass constrained is in force, it is possible to remove this assumption by using a strategy developed by Leoni in [31]. We will use this in Theorem 13.

Remark 4. Our analysis is restricted to the case of two wells. For the case of multiple wells, the result still holds after some minor modifications, by incorporating the techniques of [6].

We now introduce the functional that we will study.

Definition 5. Let $(\eta_n)_n$, $(\delta_n)_n$, $(\varepsilon_n)_n$ be infinitesimal sequences. For each $n\in\mathbb{N}$, define $F_n^{(1)}\colon L^1(\Omega;\mathbb{R}^M)\to[0,+\infty]$ as

$$F_n^{(1)}(u):=\int_\Omega\left[\frac{1}{\varepsilon_n}W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u\right)+\varepsilon_n|\nabla u|^2\right]\mathrm{d}x,$$

if $u\in W^{1,2}(\Omega;\mathbb{R}^M)$, and $+\infty$ otherwise.

In the following, we will always require $\eta_n\ll\delta_n$, that is the microscales are separated. Indeed, in the case $(\eta_n\approx\delta_n)\ll\varepsilon_n$, the study reduces to that considered in [16].

The main results are the following.

Theorem 6. *(**Compactness**) Let $(\eta_n)_n$, $(\delta_n)_n$, $(\varepsilon_n)_n$ be infinitesimal sequences such that $\eta_n\ll\delta_n\ll\varepsilon_n$, that is*

$$\lim_{n\to\infty}\frac{\eta_n}{\delta_n}=0,\qquad \lim_{n\to\infty}\frac{\delta_n}{\varepsilon_n}=0.$$

Assume that (H1), (H2), (H3), and (H5) hold. Let $(u_n)_n\subset L^1(\Omega;\mathbb{R}^M)$ be a sequence of functions such that

$$\sup_{n\in\mathbb{N}}F_n^{(1)}(u_n)<\infty.$$

Then, there exists a subsequence $(u_{n_k})_k\subset W^{1,2}(\Omega;\mathbb{R}^M)$ and a function $u\in\mathrm{BV}(\Omega;\{a,b\})$ such that $u_{n_k}\to u$ strongly in $L^1(\Omega;\mathbb{R}^M)$.

We now define the limit functional.

Definition 7. Let $u\in L^1(\Omega;\mathbb{R}^M)$. Define $F_\infty^{(1)}\colon L^1(\Omega;\mathbb{R}^M)\to[0,+\infty]$ as

$$F_\infty^{(1)}(u):=\begin{cases}\sigma^{\mathrm{h}}\,\mathrm{Per}(\{u=a\};\Omega) & u\in\mathrm{BV}(\Omega;\{a,b\}),\\ +\infty & \text{otherwise},\end{cases}\tag{15}$$

where σ^{h} is given by:

$$\sigma^{\mathrm{h}}:=\inf\left\{\int_{-1}^1 2\sqrt{W^{\mathrm{h}}(\gamma(t))}\,|\gamma'(t)|\,\mathrm{d}t:\gamma\in\mathrm{Lip}([-1,1];\mathbb{R}^M),\gamma(-1)=a,\gamma(1)=b\right\},\tag{16}$$

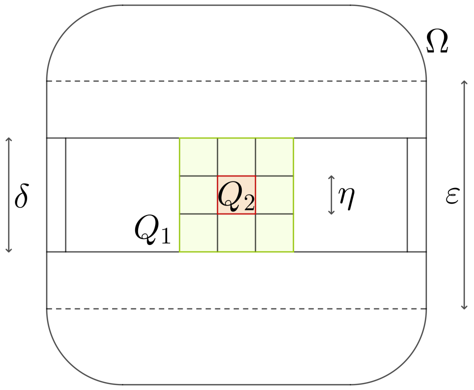

FIGURE 3. The regime considered in this paper: $\eta \ll \delta \ll \varepsilon$

and $W^{\mathrm{h}}(z)$ is defined as

$$W^{\mathrm{h}}(z) := \fint_{Q_1} \fint_{Q_2} W(y_1, y_2, z) \, \mathrm{d}y_1 \, \mathrm{d}y_2, \tag{17}$$

for all $z \in \mathbb{R}^M$.

Remark 8. It turns out that the proofs are slightly easier if, in the definition of σ^{h}, we allow γ to be in a slightly different family than $\mathrm{Lip}([-1,1];\mathbb{R}^M)$, namely $\mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M)$, which is the space of continuous curves $\gamma\colon [-1,1] \to \mathbb{R}^M$ such that $\gamma \in \mathrm{Lip}(T;\mathbb{R}^M)$ for every compact set $T \subset [-1,1]$ disjoint from $\{t \in [-1,1] : \gamma(t) \in \{a,b\}\}$.

Theorem 9. *(Γ-**convergence**) Let $(\eta_n)_n$, $(\delta_n)_n$, $(\varepsilon_n)_n$ be infinitesimal sequences such that $\eta_n \ll \delta_n \ll \varepsilon_n$, that is*

$$\lim_{n\to\infty} \frac{\eta_n}{\delta_n} = 0, \qquad \lim_{n\to\infty} \frac{\delta_n}{\varepsilon_n} = 0.$$

Assume that (H1)-(H4) hold. Then, as $n \to \infty$, $F_n^{(1)}$ Γ-converges to $F_\infty^{(1)}$ with respect to the strong $L^1(\Omega;\mathbb{R}^M)$ convergence.

Remark 10. The analysis of this paper and the following other four regimes are restricted to the case of two microscales. The type of potential that we have in mind is of the form

$$W(y_1,y_2,z) := \mathbb{1}_{I_1}(y_1)[\mathbb{1}_{I_2}(y_2)W_1(u) + \mathbb{1}_{Q_2\setminus I_2}(y_2)W_2(u)] + \mathbb{1}_{Q_1\setminus I_1}(y_1)W_3(u)$$

In the case of multiple microscales, our result still applies. In particular, it shows that in the case the parameter ε is larger than every scale of the microstructure, the surface density of the limiting functional is obtained by taking the weighted averages of the potentials.

The proofs for compactness and Γ-convergence are robust enough to be applied (with minor modifications) to the case of a mass-constrained functional.

Definition 11. (**Mass-constrained case**) Let $m \in (0,1)$. We define the mass-constrained functionals $\widehat{F}_n^{(1)} : L^1(\Omega;\mathbb{R}^M) \to [0,+\infty]$ as

$$\widehat{F}_n^{(1)}(u) := \begin{cases} F_n^{(1)}(u) & u \in W^{1,2}(\Omega;\mathbb{R}^M), \displaystyle\int_\Omega u \, \mathrm{d}x = ma + (1-m)b, \\ +\infty & \text{otherwise,} \end{cases}$$

and $\widehat{F}_\infty^{(1)} \colon L^1(\Omega;\mathbb{R}^M) \to [0,+\infty]$ as

$$\widehat{F}_\infty^{(1)}(u) := \begin{cases} F_\infty^{(1)}(u) & u \in \mathrm{BV}(\Omega;\{a,b\}), \displaystyle\int_\Omega u \, \mathrm{d}x = ma + (1-m)b, \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 12. The case $N = 1, M = 1$ has a classical proof based on traslations of the optimal profile along the real line. Here, we treat the case $N > 1$, starting from the strategy in [6, Lemma 3.3]. As far as we know, under our weak assumptions on W, there are no proofs for the case $N = 1, M > 1$.

Theorem 13. *Let $m \in (0,1)$, and let $(\eta_n)_n$, $(\delta_n)_n$, $(\varepsilon_n)_n$ be infinitesimal sequences such that $\eta_n \ll \delta_n \ll \varepsilon_n$, that is*

$$\lim_{n\to\infty} \frac{\eta_n}{\delta_n} = 0, \qquad \lim_{n\to\infty} \frac{\delta_n}{\varepsilon_n} = 0.$$

Then, the following hold:

(1) *Let $(u_n)_n \subset L^1(\Omega;\mathbb{R}^M)$ be a sequence of functions such that*

$$\sup_{n\in\mathbb{N}} \widehat{F}_n^{(1)}(u_n) < \infty.$$

Assume that (H1), (H2), (H3), and (H5) hold. Then, there exists a subsequence $(u_{n_k})_k \subset W^{1,2}(\Omega;\mathbb{R}^M)$ and a function $u \in \mathrm{BV}(\Omega;\{a,b\})$ such that

$$u_{n_k} \to u \quad \text{strongly in } L^1(\Omega;\mathbb{R}^M).$$

(2) *Assume that (H1)-(H4) hold. Then, $(\widehat{F}_n^{(1)})_n$ Γ-converges with respect to the strong $L^1(\Omega;\mathbb{R}^M)$ convergence to $\widehat{F}_\infty^{(1)}$, as $n \to \infty$.*

Remark 14. In case the potential W is assumed to be of class C^2 in the last variable, a simpler proof gives the above result for all $N, M \geq 1$ (see [29]).

2.1. **Outline of the strategy.** In this section, we explain the main novel ideas and the challenges of the proofs of the main result, Theorem 9.

For the liminf inequality, the main challenge is to first take the limit as $\delta_n \to 0$ while keeping ε_n fixed, and then sending $\varepsilon_n \to 0$. In the case of one scale of microstructure, in [16] the authors introduces a strategy to make this argument rigorous. The idea is the following: given a sequence $(u_n)_{n\in\mathbb{N}} \subset L^1(\Omega;\mathbb{R}^M)$ such that $u_n \to u$ for some $u \in BV(\Omega;\{a,b\})$, we focus on the potential term

$$\int_\Omega W\left(\frac{x}{\delta_n}, u_n(x)\right) \mathrm{d}x.$$

We would like to replace the integrand

$$W\left(\frac{x}{\delta_n}, u_n(x)\right) \qquad \text{with} \qquad W^{\mathrm{h}}\left(u_n(x)\right).$$

This, in general, requires the *technical* strong assumption $\delta \ll \varepsilon^{3/2}$ to use a Poincaré inequality to perform the pointwise substitution (see [28], and also [3]) since the above integral is multiplied by ε_n^{-1}. Since we are interested in a liminf inequality, we reason as follows. Using the unfolding operator $\mathcal{U}_1$ at scale δ_n, we can write (up to asymptotically negligible terms)

$$\int_\Omega W\left(\frac{x}{\delta_n}, u_n(x)\right) \mathrm{d}x = \int_\Omega \int_{Q_1} W(y, \mathcal{U}_1 u_n(x,y))\, \mathrm{d}y\, \mathrm{d}x.$$

If in the integrand on the right-hand side we had $u_n(x)$ in place of $\mathcal{U}_1 u_n(x,y)$, we would have done, since that term would exactly be the homogenized potential $W^{\mathrm{h}}(u_n(x))$. Having this in mind, we write

$$\int_\Omega \int_{Q_1} W(y, \mathcal{U}_1 u_n(x,y))\, \mathrm{d}y\, \mathrm{d}x = \int_\Omega \int_{Q_1} W(y, u_n(x) + (\mathcal{U}_1 u_n(x,y) - u_n(x)))\, \mathrm{d}y\, \mathrm{d}x,$$

and we treat the term $\mathcal{U}_1 u_n(x,y) - u_n(x)$ as a perturbation that we expect to vanishes as $n \to \infty$. In particular, fixed $\xi > 0$, we can consider the space of such admissible perturbations $\mathcal{A}_\xi$ that will have to satisfy some technical conditions that we do not specify in here (see [16, Definition

3.3]), but with the fundamental property that $\mathcal{A}_\xi$ contains only the zero function when $\xi = 0$. Thus, we get

$$\int_\Omega \int_{Q_1} W(y, u_n(x) + (\mathcal{U}_1 u_n(x,y) - u_n(x)))\,\mathrm{d}y\,\mathrm{d}x \geq \int_\Omega W^\xi(u_n(x))\,\mathrm{d}x, \tag{18}$$

where

$$W^\xi(p) := \inf\left\{ \int_{Q_1} W(y, p + \psi(y))\,\mathrm{d}y : \psi \in \mathcal{A}_\xi \right\}.$$

Therefore, we got rid of the scale δ_n, at the cost of introducing a slighter different potential W^ξ in place of W^{h}. This will need to be fixed at the end of the proof. Indeed, using (18), the standard argument for the liminf of the Modica-Mortola functional yields that

$$\liminf_{n\to\infty} F_n^{(1)}(u_n) \geq \sigma^\xi \operatorname{Per}(\{u = a\}; \Omega),$$

where

$$\sigma^\xi := \inf\left\{ \int_{-1}^{1} 2\sqrt{W^\xi(\gamma(t))}\,|\gamma'(t)|\,\mathrm{d}t : \gamma \in \operatorname{Lip}([-1,1]; \mathbb{R}^M), \gamma(-1) = a, \gamma(1) = b \right\},$$

Therefore, in order to conclude, we need to prove that

$$\lim_{\xi\to 0} \sigma^\xi = \sigma^{\mathrm{h}}.$$

This is done by a study of the geodesic problem defining those two quantities, using the fact that W^ξ converges to W^{hom} locally uniformly.

For multiple scales, one would expect a similar strategy to work. One of the main contribution of the present paper is the formalization of the argument and the carrying out the delicate analysis to prove that all the errors introduced by the approximation vanishes in the limit. Indeed, the technicalities involved in the several steps of the proof are far from being a trivial adaptation of the argument for the one scale case. In particular, if for a single scale, the periodic unfolding is used, for two scales, a *double periodic unfolding* $\mathcal{U}_2$ is needed. This mathematical tool has been recently developed by Damlamian, Griso, and Cioranescu in [14] (see next section for more details) and allows to write

$$\int_\Omega W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n(x)\right)\,\mathrm{d}x = \int_\Omega \int_{Q_1} \int_{Q_2} W\left(y_1, y_2, \mathcal{U}_2 u_n(x,y)\right)\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x.$$

In order to separate the contribution of two scales η_n and δ_n, we write

$$\mathcal{U}_2 u_n(x,y) = u_n(x) + [\mathcal{U}_1 u_n(x,y) - u_n(x)] + [\mathcal{U}_2 u_n(x,y) - \mathcal{U}_1 u_n(x,y)]. \tag{19}$$

This is a source of technical difficulties. First of all, we needed to identify the right classes of admissible competitors for the infimum problem that defines the approximate functional W^ξ (see Definition 33); it turns out that what makes the analysis work is to have admissible classes that also include some pointwise conditions. Unfortunately, the terms on the right-hand side of (19) that are not $u_n(x)$ do not actually belong to the required space of admissible competitors. Therefore, we need to estimate that the region where all of the conditions are satisfied is, asymptotically, of full measure. Finally, checking that the energy density σ^ξ for the approximate homogenized potential W^ξ converge to σ^{h}, requires a careful analysis, since in the case of multiple scales, the wells of W^ξ are not single wells anymore, but balls centered at the wells a and b.

We note that, in reviewing the strategy for the one scale case, we are able to simplify several steps of the proof of [16, Theorem 4.1].

The construction of the limsup inequality follows a standard approximation argument, where we reduce ourselves to defining it essentially only for the case of a single flat interface. Since the quantity that we want to approximate is σ^{h}, we take an approximate geodesics for the problem defining it, and we rescaled such a curve in the normal direction of the interface in a tubular

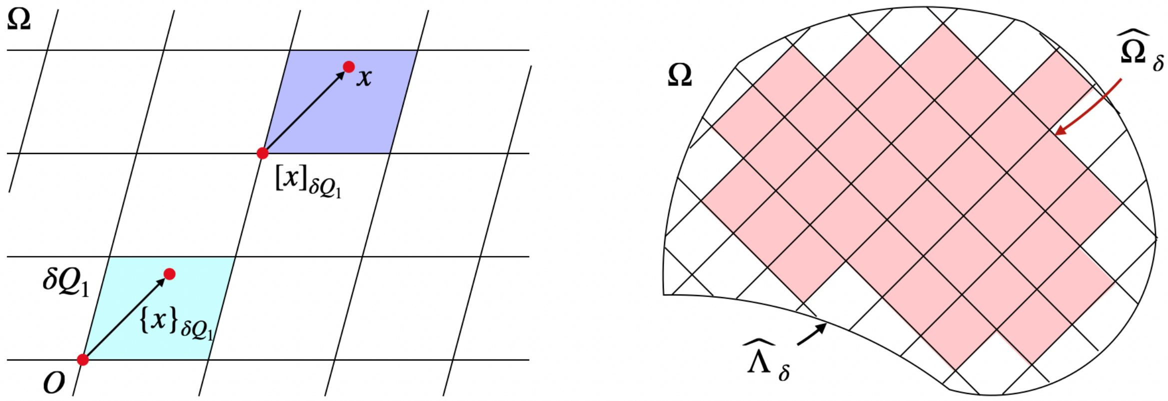

FIGURE 4. Left: Idea behind the single scale unfolding. Right: Decomposition into a proper set and a boundary set.

neighborhood of size ε_n. The technical difficulties now lies in checking that, for this particular sequence $(u_n)_n$ it holds that

$$\lim_{n\to\infty}\left|\int_\Omega W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n(x)\right)-W^{\mathrm{h}}(u_n(x))\right|=0.$$

This is essentially based on a continuity argument for W and for the function u_n.

3. Preliminaries

3.1. **Unfolding operator.** We recall the classical notion of single scale Unfolding Operator defined in [13] and recall the new notion of double scale unfolding operator, as defined [14]. Let $\Omega\in\mathbb{R}^N$ be a bounded open set. Let G_δ, G_η be two subgroups of $\mathbb{R}^N$ with rank N, corresponding to the microscales δ and η. In order to make these definitions non-trivial, we require $\eta\ll\delta$, that is the microscales are separated. Let Q_1 be a periodicity cell with respect to G_δ, and let Q_2 be a periodicity cell with respect to G_η. We assume them to be bounded, with Lipschitz boundary, and containing the origin. Points in Q_1, Q_2 will be denoted respectively by y_1, y_2.

Definition 15. (**First unfolding operator**) Let us define:

- $\Xi_1 \coloneqq \{\xi\in G_1 : \delta(\xi+Q_1)\subset\Omega\}$,
- $\widehat{\Omega}_\delta \coloneqq \displaystyle\bigcup_{\xi\in\Xi_1}\delta\left(\xi+\overline{Q}_1\right)$,
- $\Lambda_\delta \coloneqq \Omega\setminus\widehat{\Omega}_\delta$.

The *first unfolding operator* $\mathcal{U}_1\colon L^2(\Omega;\mathbb{R}^M)\to L^2(\Omega;L^2(Q_1;\mathbb{R}^M))$ is defined as:

$$\mathcal{U}_1\phi(x,y_1)\coloneqq\begin{cases}\phi\left(\delta\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1}+\delta y_1\right) & \text{a.e. in } \widehat{\Omega}_\delta\times Q_1,\\ a & \text{a.e. in } \Lambda_\delta\times Q_1,\end{cases}$$

where $\lfloor z\rfloor_{Q_1}\in G_1$ and $\{z\}_{Q_1}\in Q_1$, such that $z=\lfloor z\rfloor_{Q_1}+\{z\}_{Q_1}$.

Definition 16. (**Second partial unfolding operator**) Let us define:

- $\Xi_2 \coloneqq \left\{\xi\in G_2 : \dfrac{\eta}{\delta}(\xi-\iota_{2,\eta}+Q_2)\subset Q_1\right\}$,
- $\widehat{Q}_{1,\eta} \coloneqq \displaystyle\bigcup_{\xi\in\Xi_2}\frac{\eta}{\delta}\left(\xi-\iota_{2,\eta}+\overline{Q}_2\right)$,
- $\Lambda_{1,\eta} \coloneqq Q_1\setminus\widehat{Q}_{1,\eta}$.

The *second partial unfolding operator* $\mathcal{U}_{2,\eta}\colon L^2(Q_1;\mathbb{R}^M)\to L^2(Q_1;L^2(Q_2;\mathbb{R}^M))$ is defined as:

$$\mathcal{U}_{2,\eta}\phi(y_1,y_2):=\begin{cases}\phi\left(\frac{\eta}{\delta}\left\lfloor\frac{\delta y_1}{\eta}\right\rfloor_{Q_2}-\frac{\eta}{\delta}\iota_{2,\eta}+\frac{\eta}{\delta}y_2\right) & \text{a.e. in } \widehat{Q}_{1,\eta}\times Q_2\\ a & \text{a.e. in } \Lambda_{1,\eta}\times Q_2\end{cases}$$

where $(\iota_{2,\eta})_\eta$ is a sequence of vectors in $\overline{Q_2}$. This is needed to account for the mismatch between the two nested microscales.

Definition 17. (**Second unfolding operator**) Let us define the second unfolding operator $\mathcal{U}_2\colon L^2(\Omega;\mathbb{R}^M)\to L^2(\Omega;L^2(Q_1;L^2(Q_2;\mathbb{R}^M)))$, as $\mathcal{U}_2:=\mathcal{U}_{2,\eta}\circ\mathcal{U}_1$, or in formulas:

$$\mathcal{U}_2\phi(x,y_1,y_2):=\begin{cases}\phi\left(\delta\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1}+\eta\left\lfloor\frac{\delta y_1}{\eta}\right\rfloor_{Q_2}-\eta\iota_{2,\eta}+\eta y_2\right) & \text{a.e. in } \widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta}\times Q_2,\\ a & \text{a.e. in } (\Omega\times Q_1)\setminus(\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta})\times Q_2,\end{cases}$$

where

$$\iota_{2,\eta}:=\left\{\frac{\delta}{\eta}\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1}\right\}_{Q_2}\in\overline{Q_2}.$$

This particular choice will be clear from later computations.

Remark 18. The presence of term $\iota_{2,\eta}$ may seem weird: it turns out that unless $G_1\subseteq G_2$ and δ/η is an integer, this term is necessary for the needed properties to hold. Its importance will be made clear in the next Lemma 19.

These definitions are slightly different from the classical ones, as in this case the operators are non-zero on the boundary sets. This allows us to simplify some of the computations.

Lemma 19. *Let $W\colon\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^M\to[0,+\infty)$ and $u\in L^2(\Omega;\mathbb{R}^M)$ be as in Theorem 9. Then we have*

$$\int_\Omega W\left(\frac{x}{\delta},\frac{x}{\eta},u\right)\mathrm{d}x\geq\int_\Omega\int_{Q_1}\int_{Q_2}W\left(y_1,y_2,\mathcal{U}_2u\right)\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x. \tag{20}$$

Proof. Using the definition of the first unfolding operator $\mathcal{U}_1$ we have

$$\begin{aligned}\int_\Omega W\left(\frac{x}{\delta},\frac{x}{\eta},u\right)\mathrm{d}x&\geq\int_{\widehat{\Omega}_\delta}W\left(\frac{x}{\delta},\frac{x}{\eta},u\right)\mathrm{d}x\\&=\sum_{\xi_1\in\Xi_1}\delta^N\int_{Q_1}W\left(y_1,\frac{\delta}{\eta}\xi_1+\frac{\delta}{\eta}y_1,u\left(\delta\xi_1+\delta y_1\right)\right)\mathrm{d}y_1\\&=\sum_{\xi_1\in\Xi_1}\int_{\delta\xi_1+\delta Q_1}\int_{Q_1}W\left(y_1,\frac{\delta}{\eta}\xi_1+\frac{\delta}{\eta}y_1,u\left(\delta\xi_1+\delta y_1\right)\right)\mathrm{d}y_1\,\mathrm{d}x,\end{aligned}$$

where we used the change of coordinates $x=\delta\xi+\delta y_1$ and used the Q_1-periodicity of W in its first argument. We now repeat this estimate using the second partial unfolding operator:

$$\begin{aligned}&\sum_{\xi_1\in\Xi_1}\int_{\delta\xi_1+\delta Q_1}\int_{Q_1}W\left(y_1,\frac{\delta}{\eta}\xi_1+\frac{\delta}{\eta}y_1,u\left(\delta\xi_1+\delta y_1\right)\right)\mathrm{d}y_1\,\mathrm{d}x\\&\geq\sum_{\xi_1\in\Xi_1}\int_{\delta\xi_1+\delta\widehat{Q}_{1,\eta}}\int_{\widehat{Q}_{1,\eta}}W\left(y_1,\frac{\delta}{\eta}\xi_1+\frac{\delta}{\eta}y_1,u\left(\delta\xi_1+\delta y_1\right)\right)\mathrm{d}y_1\,\mathrm{d}x\\&=\sum_{\xi_1\in\Xi_1}\int_{\delta\xi_1+\delta\widehat{Q}_{1,\eta}}\sum_{\xi_2\in\Xi_2}\left(\frac{\eta}{\delta}\right)^N\int_{Q_2}W\left(\frac{\eta}{\delta}\xi_2-\frac{\eta}{\delta}\iota_{2,\eta}+\frac{\eta}{\delta}y_2,y_2,u\left(\delta\xi_1+\eta\xi_2-\eta\iota_{2,\eta}+\eta y_2\right)\right)\mathrm{d}y_2\,\mathrm{d}x\\&=:I,\end{aligned}$$

where

$$\iota_{2,\eta}=\left\{\frac{\delta}{\eta}\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1}\right\}_{Q_2}\equiv\frac{\delta}{\eta}\xi_1\quad\text{on } Q_2.$$

We used the change of coordinates $y_1 = \frac{\eta}{\delta} - \frac{\eta}{\delta}\iota_{2,\eta} + \frac{\eta}{\delta}y_2$. From this it is clear why we chose this particular value of $\iota_{2,\eta}$: substituting the change of coordinates in the second argument of W we get

$$\frac{\delta}{\eta}\xi_1 + \frac{\delta}{\eta}y_1 = \frac{\delta}{\eta}\xi_1 + \xi_2 + y_2 - \frac{\delta}{\eta}\xi_1 = \xi_2 + y_2 = y_2 \quad \text{on } Q_2.$$

Therefore, we can continue with the estimate:

$$\begin{aligned}
I &= \sum_{\xi_1\in\Xi_1}\int_{\delta\xi_1+\delta\widehat{Q}_{1,\eta}}\sum_{\xi_2\in\Xi_2}\left(\frac{\eta}{\delta}\right)^N\int_{Q_2} W\left(\frac{\eta}{\delta}\xi_2 - \frac{\eta}{\delta}\iota_{2,\eta} + \frac{\eta}{\delta}y_2, y_2, u\left(\delta\xi_1 + \eta\xi_2 - \eta\iota_{2,\eta} + \eta y_2\right)\right)\mathrm{d}y_2\,\mathrm{d}x\\
&= \sum_{\xi_1\in\Xi_1}\sum_{\xi_2\in\Xi_2}\int_{\delta\xi_1+\delta\widehat{Q}_{1,\eta}}\int_{\frac{\eta}{\delta}\xi_2-\frac{\eta}{\delta}\iota_{2,\eta}+\frac{\eta}{\delta}Q_2}\int_{Q_2} W\left(y_1, y_2, u\left(\delta\xi_1 + \eta\xi_2 - \eta\iota_{2,\eta} + \eta y_2\right)\right)\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x\\
&= \int_{\widehat{\Omega}_\delta}\int_{\widehat{Q}_{1,\eta}}\int_{Q_2} W\left(y_1, y_2, \mathcal{U}_2 u\right)\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x,
\end{aligned}$$

where in the last equality we used that

$$\forall x \in \delta\xi_1 + \delta\widehat{Q}_{1,\eta}, \qquad \left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1} = \left\lfloor\xi_1 + \widehat{Q}_{1,\eta}\right\rfloor_{Q_1} = \xi_1,$$

and also that

$$\forall y_1 \in \frac{\eta}{\delta}\xi_2 - \frac{\eta}{\delta}\left\{\frac{\delta}{\eta}\xi_1\right\}_{Q_2} + \frac{\eta}{\delta}Q_2, \qquad \left\lfloor\frac{\delta y_1}{\eta}\right\rfloor_{Q_2} = \left\lfloor\xi_2 - \left\{\frac{\delta}{\eta}\xi_1\right\}_{Q_2} + Q_2\right\rfloor_{Q_2} = \xi_2,$$

which give the exact definition of the second unfolding unfolding operator on $\widehat{\Omega}_\delta \times \widehat{Q}_{1,\eta} \times Q_2$. Now, since we have defined the unfolding operator to be equal to a on the boundary sets, we have that

$$\int_{(\Omega\times Q_1)\setminus(\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta})\times Q_2} W(y_1, y_2, \mathcal{U}_2 u)\,\mathrm{d}y_2\mathrm{d}y_1\mathrm{d}x = \int_{(\Omega\times Q_1)\setminus(\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta})\times Q_2} W(y_1, y_2, a)\,\mathrm{d}y_2\mathrm{d}y_1\mathrm{d}x = 0.$$

This lets us add back the boundary set to the integral, therefore having:

$$\int_{\widehat{\Omega}_\delta}\int_{\widehat{Q}_{1,\eta}}\int_{Q_2} W\left(y_1, y_2, \mathcal{U}_2 u\right)\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x = \int_{\Omega}\int_{Q_1}\int_{Q_2} W\left(y_1, y_2, \mathcal{U}_2 u\right)\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x.$$

Summing up everything, we have that

$$\int_\Omega W\left(\frac{x}{\delta}, \frac{x}{\eta}, u\right)\mathrm{d}x \geq \int_{\Omega}\int_{Q_1}\int_{Q_2} W\left(y_1, y_2, \mathcal{U}_2 u\right)\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x.$$

□

Remark 20. As is shown from these computations, this result does not change if we choose another value for the first unfolding operator $\mathcal{U}_1$ on the boundary set, as long as it is a constant. The only important choice is $\mathcal{U}_2 u = a$ on the boundary set.

The following propositions will be important later; for their proofs we refer the reader to [13, 14].

Proposition 21. *Let $u \in L^2(\Omega;\mathbb{R}^M)$. Then:*

(1) the first unfolding operator $\mathcal{U}_1$ is linear, continuous and bounded from $L^2(\Omega;\mathbb{R}^M)$ to $L^2(\Omega; L^2(Q_1;\mathbb{R}^M))$;
(2) the second unfolding operator $\mathcal{U}_2$ is linear, continuous and bounded from $L^2(\Omega;\mathbb{R}^M)$ to $L^2(\Omega; L^2(Q_1; L^2(Q_2;\mathbb{R}^M)))$.

Moreover, if $u \in W^{1,2}(\Omega;\mathbb{R}^M)$, the chain rule holds for both unfolding operators, therefore:

$$\|\nabla_{y_1}\mathcal{U}_1 u\|_{L^2(\Omega;L^2(Q_1;\mathbb{R}^{N\times M}))} = \delta\|\mathcal{U}_1\nabla u\|_{L^2(\Omega;L^2(Q_1;\mathbb{R}^{N\times M}))} \leq \delta\|\nabla u\|_{L^2(\Omega;\mathbb{R}^{N\times M})}, \tag{21}$$

$$\|\nabla_{y_2}\mathcal{U}_2 u\|_{L^2(\Omega;L^2(Q_1;L^2(Q_2;\mathbb{R}^{N\times M})))} = \eta\|\mathcal{U}_2\nabla u\|_{L^2(\Omega;L^2(Q_1;L^2(Q_2;\mathbb{R}^{N\times M})))} \leq \eta\|\nabla u\|_{L^2(\Omega;\mathbb{R}^{N\times M})}. \tag{22}$$

Proposition 22. *Let us define $\mathcal{G}_1\colon \Omega\times Q_1\to\Omega$ and $\mathcal{G}_2\colon\Omega\times Q_1\times Q_2\to\Omega$ as*

$$\mathcal{G}_1(x,y_1) \coloneqq \delta\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1} + \delta y_1,$$
$$\mathcal{G}_2(x,y_1,y_2) \coloneqq \delta\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1} + \eta\left\lfloor\frac{\delta y_1}{\eta}\right\rfloor_{Q_2} - \eta\iota_{2,\eta} + \eta y_2,$$

where $\iota_{2,\eta}$ is chosen as above. This then implies:

$$\mathcal{U}_1 u = u\circ\mathcal{G}_1 \quad \text{a.e. in } \widehat{\Omega}_\delta\times Q_1;$$
$$\mathcal{U}_2 u = u\circ\mathcal{G}_2 \quad \text{a.e. in } \widehat{\Omega}_\delta\times \widehat{Q}_{1,\eta}\times Q_2.$$

Then the following hold:

(i) $|\mathcal{G}_2(x,y_1,y_2)-\mathcal{G}_1(x,y_1)|\le c\eta$;

(ii) $|\mathcal{G}_2(x,y_1,y_2)-x|\le c\delta$.

Proposition 23. *Let $\phi\in L^1(\Omega;\mathbb{R}^M)$, and let $v,w\in L^2(\Omega;\mathbb{R}^M)$. Let $\widehat{\Omega}_1$ be the image of $\widehat{\Omega}_\delta\times Q_1$ under the map $\mathcal{G}_1$, and let $\widehat{\Omega}_2$ be the image of $\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta}\times Q_2$ under the map $\mathcal{G}_2$.*
Then the following hold:

(i) $\mathcal{U}_1(vw)=\mathcal{U}_1 v\cdot\mathcal{U}_1 w$ *on* $\widehat{\Omega}_\delta\times Q_1$;

(ii) $\mathcal{U}_2(vw)=\mathcal{U}_2 v\cdot\mathcal{U}_2 w$ *on* $\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta}\times Q_2$;

(iii) $\displaystyle\int_\Omega\int_{Q_1}\mathcal{U}_1\phi(x,y_1)\,\mathrm{d}y_1\,\mathrm{d}x = \int_{\widehat{\Omega}_1}\phi(x)\,\mathrm{d}x$;

(iv) $\displaystyle\int_\Omega\int_{Q_1\times Q_2}\mathcal{U}_2\phi(x,y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}x = \int_{\widehat{\Omega}_2}\phi(x)\,\mathrm{d}x$;

(v) $\displaystyle\left|\int_\Omega\int_{Q_1}\mathcal{U}_1\phi(x,y_1)\,\mathrm{d}y_1\,\mathrm{d}x - \int_\Omega\phi(x)\,\mathrm{d}x\right| \le \int_{\Omega\setminus\widehat{\Omega}_1}|\phi|(x)\,\mathrm{d}x$;

(vi) $\displaystyle\left|\int_\Omega\int_{Q_1\times Q_2}\mathcal{U}_2\phi(x,y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}x - \int_\Omega\phi(x)\,\mathrm{d}x\right| \le \int_{\Omega\setminus\widehat{\Omega}_2}|\phi|(x)\,\mathrm{d}x.$

Proposition 24. *Let $w\in L^2(\Omega;\mathbb{R}^M)$. Then the following hold:*

(i) $\mathcal{U}_1 w\to w$ *strongly in* $L^2(\Omega;L^2(Q_1;\mathbb{R}^M))$ *as* $\delta\to0$;

(ii) $\mathcal{U}_2 w\to w$ *strongly in* $L^2(\Omega;L^2(Q_1;L^2(Q_2;\mathbb{R}^M)))$ *as* $\eta,\delta\to0$.

Proposition 25. *Let $(w_\delta)_{\delta\to0}\subset L^2(\Omega;\mathbb{R}^M)$ be a sequence converging strongly to $w_1\in L^2(\Omega;\mathbb{R}^M)$ as $\delta\to0$, and let $(w_\eta)_{\eta\to0}\subset L^2(\Omega;\mathbb{R}^M)$ be a sequence converging strongly to $w_2\in L^2(\Omega;\mathbb{R}^M)$ as $\eta,\delta\to0$. Then the following hold:*

(i) $\mathcal{U}_1 w_\delta\to w_1$ *strongly in* $L^2(\Omega;L^2(Q_1;\mathbb{R}^M))$ *as* $\delta\to0$;

(ii) $\mathcal{U}_2 w_\eta\to w_2$ *strongly in* $L^2(\Omega;L^2(Q_1;L^2(Q_2;\mathbb{R}^M)))$ *as* $\eta,\delta\to0$.

3.2. **Γ-convergence.** In this section, we recall the definition and the basic properties of Γ-limits. Since in this paper we work in the setting of the metric space $L^1(\Omega;\mathbb{R}^M)$, we will present the equivalent definition with sequences. We refer to [19] (see also [9]) for a complete study of Γ-convergence on topological spaces.

Definition 26. Let (X,d) be a metric space, and let $(F_n)_n$ be a sequence of functionals $F_n: X\to[-\infty,+\infty]$. We say that $(F_n)_n$ Γ-converges to $F: X\to[-\infty,+\infty]$ with respect to the metric d, if the following hold:

(i) (Liminf inequality) For every $x\in X$ and every $(x_n)_n\subset X$ with $x_n\to x$, we have

$$F(x)\le\liminf_{n\to\infty}F_n(x_n),$$

(ii) (Limsup inequality) For every $x \in X$, there exists $(x_n)_n \subset X$ such that

$$\limsup_{n\to\infty} F_n(x_n) \leq F(x),$$

and with $x_n \to x$.

The notion of Γ-convergence was designed to characterize in a variational way the limiting behavior of sequences of global minimizers, as well as of the minima (see, for example, [19, Corollary 7.20]).

Theorem 27. *Let (X, d) be a metric space. Consider, for each $n \in \mathbb{N}$, a functional $F_n : X \to \mathbb{R} \cup \{\infty\}$, and assume that the sequence $(F_n)_n$ Γ-converges to some $F : X \to \mathbb{R} \cup \{\infty\}$. For each $n \in \mathbb{N}$, let $x_n \in X$ be a minimizer of F_n on X. Then, every cluster point $x \in X$ of $(x_n)_n$ is a minimizer of F, and*

$$F(x) = \limsup_{n\to\infty} F_n(x_n).$$

If the point $x \in X$ is a limit of the sequence $(x_n)_n$, then the above limsup is actually a limit.

3.3. **Sets of finite perimeter.** We recall the definition and some basic facts about sets of finite perimeter that are needed in the paper. For more details on the subject, we refer the reader to standard references, such as [1, 24, 27, 32].

Definition 28. Let $E \subset \mathbb{R}^N$ with $|E| < \infty$, and let $A \subset \mathbb{R}^N$ be an open set. We say that E has *finite perimeter* in A if

$$P(E; A) := \sup\left\{ \int_E \mathrm{div}\varphi \, dx \, : \, \varphi \in C^1_c(A; \mathbb{R}^N) \, , \, \|\varphi\|_{L^\infty} \leq 1 \right\} < \infty.$$

Definition 29. Let $a, b \in \mathbb{R}^M$. We define the space $BV(\Omega; \{a, b\})$ as the space of functions $u \in L^1(\Omega; \mathbb{R}^M)$ with $u(x) \in \{a, b\}$ for a.e. $x \in \Omega$, and such that the set $\{x \in \Omega : u(x) = a\}$ has finite perimeter in Ω.

Definition 30. Let $E \subset \mathbb{R}^N$ be a set of finite perimeter in the open set $A \subset \mathbb{R}^N$. We define $\partial^* E$, the *reduced boundary* of E, as the set of points $x \in \mathbb{R}^N$ for which the limit

$$\nu_E(x) := -\lim_{r\to 0} \frac{D\chi_E(B(x, r))}{|D\chi_E|(B(x, r))}$$

exists and is such that $|\nu_E(x)| = 1$. The vector $\nu_E(x)$ is called the *measure theoretic exterior normal* to E at x.

We recall part of the De Giorgi's structure theorem for sets of finite perimeter.

Theorem 31. *Let $E \subset \mathbb{R}^N$ be a set of finite perimeter in the open set $A \subset \mathbb{R}^N$. Then,*

$$P(E, B) = \mathcal{H}^{N-1}(\partial^* E \cap B),$$

for all Borel sets $B \subset A$.

4. Technical results

4.1. Estimates for sequences with uniformly bounded energies.

Let $(u_n)_n \subset W^{1,2}(\Omega; \mathbb{R}^M)$ be a sequence such that

$$\sup_n F^{(1)}_n(u_n) = C < +\infty.$$

Then,

$$\|\nabla u_n\|^2_{L^2(\Omega;\mathbb{R}^{N\times M})} = \int_\Omega |\nabla u_n|^2 \, \mathrm{d}x = \frac{1}{\varepsilon_n}\int_\Omega \varepsilon_n |\nabla u_n|^2 \, \mathrm{d}x \leq \frac{1}{\varepsilon_n} F^{(1)}_n(u_n) \leq \frac{C}{\varepsilon_n}.$$

Using the chain rule from (21) and from (22) we can therefore deduce that

$$\|\nabla_{y_1} \mathcal{U}_1 u_n\|^2_{L^2(\Omega;L^2(Q_1;\mathbb{R}^{N\times M}))} \leq \delta_n^2 \|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^{N\times M})} \leq C\frac{\delta_n^2}{\varepsilon_n}, \tag{23}$$

$$\|\nabla_{y_2}\mathcal{U}_2 u_n\|^2_{L^2(\Omega;L^2(Q_1\times Q_2;\mathbb{R}^{N\times M}))} \le \eta_n^2\|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^{N\times M})} \le C\frac{\eta_n^2}{\varepsilon_n}. \tag{24}$$

We will now state a theorem that is needed for a key-step of the proof. We will only give the proof for the second formula involving the two-scale unfolding operator, and we refer to [16, Theorem 3.2] for the proof of the first one.

Theorem 32. *Let $(u_n)_n \subset W^{1,2}(\Omega;\mathbb{R}^M)$ such that*

$$\sup_n F_n^{(1)}(u_n) = C < +\infty, \qquad \sup_n \|u_n\|_\infty \le M < +\infty.$$

Then, it holds that

$$\|\mathcal{U}_1 u_n - u_n\|^2_{L^2(\Omega;L^2(Q_1;\mathbb{R}^M))} \le C\delta_n, \tag{25}$$

$$\|\mathcal{U}_2 u_n - \mathcal{U}_1 u_n\|^2_{L^2(\Omega;L^2(Q_1;L^2(Q_2;\mathbb{R}^M)))} \le C\frac{\eta_n}{\delta_n}. \tag{26}$$

Proof. As the proof of (25) involves the same steps as the proof of (26) but with less details, we will only focus on the latter, and refer to [16, Theorem 3.2] for the proof of the former.
For $x \in \Omega, y_1 \in Q_1$, we define

$$(\mathcal{U}_2 u_n)_{Q_2}(x, y_1) := \int_{Q_2} \mathcal{U}_2 u_n(x, y_1, y_2)\, \mathrm{d}y_2.$$

Using the triangle inequality, we rewrite

$$\begin{aligned}\|\mathcal{U}_2 u_n - \mathcal{U}_1 u_n\|^2_{L^2} &= \|\mathcal{U}_2 u_n - (\mathcal{U}_2 u_n)_{Q_2} + (\mathcal{U}_2 u_n)_{Q_2} - \mathcal{U}_1 u_n\|^2_{L^2(\Omega;L^2(Q_1\times Q_2;\mathbb{R}^M))} \\ &\le 2\|\mathcal{U}_2 u_n - (\mathcal{U}_2 u_n)_{Q_2}\|^2_{L^2(\Omega;L^2(Q_1\times Q_2;\mathbb{R}^M))} \qquad (27)\\ &\quad + 2\|(\mathcal{U}_2 u_n)_{Q_2} - \mathcal{U}_1 u_n\|^2_{L^2(\Omega;L^2(Q_1\times Q_2;\mathbb{R}^M))}. \qquad (28)\end{aligned}$$

We first estimate the first term on the right-hand side of (28). Let us fix $x \in \Omega$ and $y_1 \in Q_1$. Using the Poincaré-Wirtinger inequality on Q_2 we can write

$$\int_{Q_2} |\mathcal{U}_2 u_n - (\mathcal{U}_2 u_n)_{Q_2}|^2\, \mathrm{d}y_2 \le C\int_{Q_2} |\nabla_{y_2}\mathcal{U}_2 u_n|^2\, \mathrm{d}y_2.$$

Integrating over $\Omega \times Q_1$ and using (24) we get

$$\|\mathcal{U}_2 u_n - (\mathcal{U}_2 u_n)_{Q_2}\|^2_{L^2(\Omega;L^2(Q_1\times Q_2;\mathbb{R}^M))} \le C\|\nabla_{y_2}\mathcal{U}_2 u_n\|^2_{L^2(\Omega;L^2(Q_1\times Q_2;\mathbb{R}^{N\times M}))} \le C\frac{\eta_n^2}{\varepsilon_n}. \tag{29}$$

We now estimate the second term on the right-hand side of (28). We get

$$\begin{aligned}&\int_{\Omega\times Q_1} |\mathcal{U}_1 u_n - (\mathcal{U}_2 u_n)_{Q_2}|^2\, \mathrm{d}y_1\, \mathrm{d}x \\ &\qquad = \int_{\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta}} |\mathcal{U}_1 u_n - (\mathcal{U}_2 u_n)_{Q_2}|^2\, \mathrm{d}y_1\, \mathrm{d}x + \int_{(\Omega\times Q_1)\setminus(\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta})} |\mathcal{U}_1 u_n - a|^2\, \mathrm{d}y_1\, \mathrm{d}x. \qquad (30)\end{aligned}$$

We estimate now the second term of the right-hand of (30). We have

$$(\Omega\times Q_1)\setminus(\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta}) = (\Lambda_\delta\times Q_1)\cup(\widehat{\Omega}_\delta\times\Lambda_{1,\eta}).$$

Since $\mathcal{U}_1 u_n = a$ on $\Lambda_\delta \times Q_1$, we get

$$\int_{(\Omega\times Q_1)\setminus(\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta})} |\mathcal{U}_1 u_n - a|^2\, \mathrm{d}y_1\, \mathrm{d}x = \int_{\widehat{\Omega}_\delta\times\Lambda_{1,\eta}} |\mathcal{U}_1 u_n - a|^2\, \mathrm{d}y_1\, \mathrm{d}x \le c\frac{\eta_n}{\delta_n}, \tag{31}$$

where the last step follows from the fact that u_n, and therefore also $\mathcal{U}_1 u_n$, is uniformly bounded in L^∞.
We now estimate the first term of the right-hand side of (30), and for this we need to use the partial unfolding $\mathcal{U}_{2,\eta}$. By definition of $\mathcal{U}_2$ we have that

$$\mathcal{U}_{2,\eta}\mathcal{U}_1 u_n = \mathcal{U}_2 u_n. \tag{32}$$

We now prove that
$$\mathcal{U}_{2,\eta}(\mathcal{U}_2u_n)_{Q_2} = (\mathcal{U}_2u_n)_{Q_2}. \tag{33}$$
To do this, it is a matter of a tedious but easy computation:

$$\begin{aligned}
&\mathcal{U}_{2,\eta}(\mathcal{U}_2u_n)_{Q_2}(x,y_1,y_2)\\
&= (\mathcal{U}_2u_n)_{Q_2}\left(x, \frac{\eta}{\delta}\left\lfloor \frac{\delta y_1}{\eta}\right\rfloor_{Q_2} - \frac{\eta}{\delta}\left\{\frac{\delta}{\eta}\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1}\right\}_{Q_2} + \frac{\eta}{\delta}y_2\right)\\
&= \int_{Q_2}\mathcal{U}_2u_n\left(x, \frac{\eta}{\delta}\left\lfloor \frac{\delta y_1}{\eta}\right\rfloor_{Q_2} - \frac{\eta}{\delta}\left\{\frac{\delta}{\eta}\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1}\right\}_{Q_2} + \frac{\eta}{\delta}y_2, y_2\right)\mathrm{d}y_2\\
&= \int_{Q_2}u_n\left(\delta\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1} + \eta\left\lfloor\frac{\delta}{\eta}\left(\frac{\eta}{\delta}\left\lfloor\frac{\delta y_2}{\eta}\right\rfloor_{Q_2} - \frac{\eta}{\delta}\left\{\frac{\delta}{\eta}\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1}\right\}_{Q_2} + \frac{\eta}{\delta}y_2\right)\right\rfloor_{Q_2} - \eta\left\{\frac{\delta}{\eta}\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1}\right\}_{Q_2} + \eta y_2\right)\mathrm{d}y_2\\
&= \int_{Q_2}u_n\left(\delta\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1} + \eta\left\lfloor\left\lfloor\frac{\delta y_2}{\eta}\right\rfloor_{Q_2} - \left\{\frac{\delta}{\eta}\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1}\right\}_{Q_2} + y_2\right\rfloor_{Q_2} - \eta\left\{\frac{\delta}{\eta}\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1}\right\}_{Q_2} + \eta y_2\right)\mathrm{d}y_2\\
&= \int_{Q_2}u_n\left(\delta\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1} + \eta\left\lfloor\frac{\delta y_2}{\eta}\right\rfloor_{Q_2} - \eta\left\{\frac{\delta}{\eta}\left\lfloor\frac{x}{\delta}\right\rfloor_{Q_1}\right\}_{Q_2} + \eta y_2\right)\mathrm{d}y_2\\
&= \int_{Q_2}\mathcal{U}_2u_n(x,y_1,y_2)\,\mathrm{d}y_2 = (\mathcal{U}_2u_n)_{Q_2}(x,y_1)
\end{aligned}$$

Therefore we now have:

$$\begin{aligned}
&\int_{\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta}}\left|\mathcal{U}_1u_n - (\mathcal{U}_2u_n)_{Q_2}\right|^2(x,y_1)\,\mathrm{d}y_1\,\mathrm{d}x\\
&\qquad= \int_{\widehat{\Omega}_\delta}\sum_{\xi_2\in\Xi_2}\left(\frac{\eta}{\delta}\right)^N\int_{Q_2}\left|\mathcal{U}_1u_n - (\mathcal{U}_2u_n)_{Q_2}\right|^2\left(x, \frac{\eta}{\delta}\xi_2 - \frac{\eta}{\delta}\iota_{2,\eta} + \frac{\eta}{\delta}y_2\right)\mathrm{d}y_2\,\mathrm{d}x\\
&\qquad= \int_{\widehat{\Omega}_\delta}\sum_{\xi_2\in\Xi_2}\int_{\frac{\eta}{\delta}\xi_2-\frac{\eta}{\delta}\iota_{2,\eta}+\frac{\eta}{\delta}Q_2}\int_{Q_2}\left|\mathcal{U}_1u_n - (\mathcal{U}_2u_n)_{Q_2}\right|^2\left(x, \frac{\eta}{\delta}\xi_2 - \frac{\eta}{\delta}\iota_{2,\eta} + \frac{\eta}{\delta}y_2\right)\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x\\
&\qquad= \int_{\widehat{\Omega}_\delta}\int_{\widehat{Q}_{1,\eta}}\int_{Q_2}\mathcal{U}_{2,\eta}\left|\mathcal{U}_1u_n - (\mathcal{U}_2u_n)_{Q_2}\right|^2\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x.
\end{aligned}$$

At this point we can use the linearity of the unfolding operator, together with (32) and (33) to see that on $\widehat{Q}_{1,\eta}\times Q_2$, where $\mathcal{U}_{2,\eta}$ is properly defined, we have
$$\mathcal{U}_{2,\eta}\left|\mathcal{U}_1u_n - (\mathcal{U}_2u_n)_{Q_2}\right|^2 = \left|\mathcal{U}_{2,\eta}\mathcal{U}_1u_n - \mathcal{U}_{2,\eta}(\mathcal{U}_2u_n)_{Q_2}\right|^2 = \left|\mathcal{U}_2u_n - (\mathcal{U}_2u_n)_{Q_2}\right|^2.$$
Substituting back we get
$$\begin{aligned}
\int_{\widehat{\Omega}_\delta}\int_{\widehat{Q}_{1,\eta}}\int_{Q_2}\mathcal{U}_{2,\eta}\left|\mathcal{U}_1u_n - (\mathcal{U}_2u_n)_{Q_2}\right|^2\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x &= \int_{\widehat{\Omega}_\delta}\int_{\widehat{Q}_{1,\eta}}\int_{Q_2}\left|\mathcal{U}_2u_n - (\mathcal{U}_2u_n)_{Q_2}\right|^2\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x\\
&= \int_{\Omega}\int_{Q_1}\int_{Q_2}\left|\mathcal{U}_2u_n - (\mathcal{U}_2u_n)_{Q_2}\right|^2\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x\\
&= \left\|\mathcal{U}_2u_n - (\mathcal{U}_2u_n)_{Q_2}\right\|^2_{L^2(\Omega;L^2(Q_1\times Q_2;\mathbb{R}^M))},
\end{aligned}$$
where again we used the fact that $\mathcal{U}_2u_n \equiv a$ on $(\Omega\times Q_1)\setminus(\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta})\times Q_2$. To sum up, we proved that
$$\int_{\widehat{\Omega}_\delta\times\widehat{Q}_{1,\eta}}\left|\mathcal{U}_1u_n - (\mathcal{U}_2u_n)_{Q_2}\right|^2\mathrm{d}y_1\,\mathrm{d}x = \left\|\mathcal{U}_2u_n - (\mathcal{U}_2u_n)_{Q_2}\right\|^2_{L^2(\Omega;L^2(Q_1;L^2(Q_2;\mathbb{R}^M)))}, \tag{34}$$
where in the L^2 norm we put back the boundary sets, as they add zero contribution to the integral.

Using (28), (31), (34) and (29), we get

$$\|\mathcal{U}_2 u_n - \mathcal{U}_1 u_n\|^2_{L^2(\Omega;L^2(Q_1\times Q_2;\mathbb{R}^M))} \leq c\left(\frac{\eta_n^2}{\varepsilon_n} + \frac{\eta_n}{\delta_n}\right) \leq c\frac{\eta_n}{\delta_n},$$

where the last inequality comes from $\eta_n \ll \delta_n \ll \varepsilon_n$. □

4.2. **Definition and properties of the auxiliary cell problem.** We first need to define an auxiliary cell problem that will be needed in the proof.

Definition 33. (**Auxiliary cell problem**) Define the function $W^\xi\colon \mathbb{R}^M \to [0,+\infty)$ as

$$W^\xi(z) := \inf_{\psi_1\in\mathcal{A}_1^\xi} \inf_{\psi_2\in\mathcal{A}_2^\xi} \int_{Q_1}\int_{Q_2} W\left(y_1, y_2, z + \psi_1(y_1) + \psi_2(y_1,y_2)\right) \mathrm{d}y_2\, \mathrm{d}y_1,$$

where the admissible classes $\mathcal{A}_1^\xi$ and $\mathcal{A}_2^\xi$ are given by:

$$\mathcal{A}_1^\xi := \left\{\psi_1 \in W^{1,2}(Q_1;\mathbb{R}^M) : \|\psi_1\|_{L^2(Q_1;\mathbb{R}^M)} \leq \xi, \|\nabla_{y_1}\psi_1\|_{L^2(Q_1;\mathbb{R}^{N\times M})} \leq 1\right\},$$

$$\mathcal{A}_2^\xi := \left\{\psi_2 \in L^2(Q_1;W^{1,2}(Q_2;\mathbb{R}^M)) : \|\psi_2\|_{L^2(Q_1\times Q_2;\mathbb{R}^M)} \leq \xi, \|\nabla_{y_2}\psi_2(y_1,\cdot)\|_{L^2(Q_2;\mathbb{R}^{N\times M})} \leq 1\right\}.$$

Theorem 34. *(**Properties of** W^ξ) The following hold:*

(1) For every $z \in \mathbb{R}^M$, the infimum problem defining $W^\xi(z)$ is well-defined, i.e. admits a minimizer;

(2) W^ξ is continuous;

(3) There exists $r(\xi) > 0$, with $r(\xi) \to 0$ as $\xi \to 0$, such that

$$W^\xi(z) = 0 \iff z \in \overline{B(a, r(\xi))} \cup \overline{B(b, r(\xi))};$$

(4) For each $z \in \mathbb{R}^M$, $W^\xi(z)$ converges increasingly to

$$W^{\mathrm{h}}(z) := \fint_{Q_1}\fint_{Q_2} W(y_1,y_2,z)\, \mathrm{d}y_1\, \mathrm{d}y_2$$

as $\xi \to 0$. Moreover, W^ξ converges uniformly to W^{h} on every compact set.

Proof. **Step 1: Proof of (1).** We first prove that we can reduce to competitors which are bounded in L^∞, and then prove the desired claim.

Step 1.1: Reduction to ψ being uniformly bounded in L^∞. Let $M > 0$ be big enough such that $M > R$, where $R > 0$ is the constant appearing in (H3). Let $\varphi_M\colon [0,+\infty) \to [0,1]$ be a smooth cut-off function such that

$$\varphi_M(t) \equiv 1 \quad t \in [0,M], \qquad \varphi_M(t) \equiv 0 \quad t \in [2M,+\infty).$$

We now modify W such that it is linear outside of a ball of radius $2M$: to do that, we define $\widetilde{W}_M\colon \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^M \to [0,+\infty)$ as

$$\widetilde{W}_M(y_1,y_2,z) := \varphi_M(|z|)W(y_1,y_2,z) + (1-\varphi_M(|z|))\frac{|z|}{R}.$$

We now prove that if

$$\widetilde{W^\xi}(z) = \inf_{\widetilde{\psi}_1\in\widetilde{\mathcal{A}}_1^\xi} \inf_{\widetilde{\psi}_2\in\widetilde{\mathcal{A}}_2^\xi} \int_{Q_1}\int_{Q_2} \widetilde{W}_M(y_1,y_2,z+\widetilde{\psi}_1(y_1)+\widetilde{\psi}_2(y_1,y_2))\, \mathrm{d}y_1\, \mathrm{d}y_2 \tag{35}$$

where

$$\widetilde{\mathcal{A}}_1^\xi := \left\{\psi_1 \in W^{1,2}(Q_1;\mathbb{R}^M) : \|\psi_1\|_\infty \leq M - \frac{|z|}{2}, \|\psi_1\|_{L^2(Q_1;\mathbb{R}^M)} \leq \xi, \|\nabla_{y_1}\psi_1\|_{L^2(Q_1;\mathbb{R}^{N\times M})} \leq 1\right\},$$

$$\widetilde{\mathcal{A}}_2^\xi := \left\{\psi_2 \in L^2(Q_1;W^{1,2}(Q_2;\mathbb{R}^M)) : \|\psi_2\|_\infty \leq M - \frac{|z|}{2},\right.$$

$$\|\psi_2\|_{L^2(Q_1\times Q_2;\mathbb{R}^M)} \leq \xi, \|\nabla_{y_2}\psi_2(y_1,\cdot)\|_{L^2(Q_2;\mathbb{R}^{N\times M})} \leq 1\Big\},$$

then $\widetilde{W}^\xi(z) \leq W^\xi(z)$. We now need to define the truncation operator $\mathcal{T}_M\colon W^{1,2}(\Omega;\mathbb{R}^M) \to L^\infty(\Omega;\mathbb{R}^M)\cap W^{1,2}(\Omega;\mathbb{R}^M)$, defined as

$$\mathcal{T}_M u(x) := \begin{cases} u(x) & |u(x)| \leq M, \\ M\frac{u(x)}{|u(x)|} & |u(x)| > M. \end{cases} \tag{36}$$

Let us now take two competitors for the original problem, $\psi_1 \in \mathcal{A}_1^\xi$ and $\psi_2 \in \mathcal{A}_2^\xi$. We define the truncated versions of these by

$$\widetilde{\psi}_1 := \mathcal{T}_{M-\frac{|z|}{2}}\psi_1 \in \widetilde{\mathcal{A}}_1^\xi, \qquad \widetilde{\psi}_2 := \mathcal{T}_{M-\frac{|z|}{2}}\psi_2 \in \widetilde{\mathcal{A}}_2^\xi,$$

which are easily seen to be competitors to the modified problem. This choice of truncation with $M-\frac{|z|}{2}$ will not be a problem, as in the liminf we will also require u_n to be uniformly bounded. Therefore, using (35) and (H3), we have

$$\begin{aligned} W^\xi(z) &\leq \int_{Q_1}\int_{Q_2} \widetilde{W}_M(y_1,y_2,z+\widetilde{\psi}_1(y_1)+\widetilde{\psi}_2(y_1,y_2))\,\mathrm{d}y_1\,\mathrm{d}y_2 \\ &\leq \int_{Q_1}\int_{Q_2} \widetilde{W}_M(y_1,y_2,z+\psi_1(y_1)+\psi_2(y_1,y_2))\,\mathrm{d}y_1\,\mathrm{d}y_2 \\ &\leq \int_{Q_1}\int_{Q_2} W(y_1,y_2,z+\psi_1(y_1)+\psi_2(y_1,y_2))\,\mathrm{d}y_1\,\mathrm{d}y_2. \end{aligned}$$

Taking the infimum over $\psi_1 \in \mathcal{A}_1^\xi$ and $\psi_2 \in \mathcal{A}_2^\xi$, we obtain the definition of the original problem, and we get $\widetilde{W}^\xi(z) \leq W^\xi(z)$. This implies that we can always lower the energy by considered this modified auxiliary problem with ψ_1 and ψ_2 bounded in L^∞. This is enough for the proofs, as the purpose of the auxiliary problem is exactly to lower the energy. Therefore, from now on we can assume ψ_1,ψ_2 to be bounded.

Step 1.2: Existence of a minimizer. Fix $z\in\mathbb{R}^M$, and let $(\psi_1^n)_n \subset \mathcal{A}_1^\xi$ and $(\psi_2^n)_n \subset \mathcal{A}_2^\xi$ two sequences dependent on z, such that

$$\int_{Q_1}\int_{Q_2} W(y_1,y_2,z+\psi_1^n(y_1)+\psi_2^n(y_1,y_2))\,\mathrm{d}y_2\,\mathrm{d}y_1 \to W^\xi(z) \qquad n\to\infty.$$

By definition of $\mathcal{A}_1^\xi$ and $\mathcal{A}_2^\xi$ we know that $(\psi_1^n)_n$ and $(\psi_2^n(y_1,\cdot))_n$ are bounded in L^2 for a.e. $y_1 \in Q_1$, therefore up to a subsequence they have weak limits, respectively $\psi_1^0 \in L^2(Q_1;\mathbb{R}^M)$ and $\psi_2^0 \in L^2(Q_1;L^2(Q_2;\mathbb{R}^M))$. The sequences are also bounded in $W^{1,2}$, therefore we also have $\psi_1^0 \in W^{1,2}(Q_1;\mathbb{R}^M)$ and $\psi_2^0(y_1,\cdot) \in W^{1,2}(Q_2;\mathbb{R}^M)$ for a.e. $y_1\in Q_1$.

In order to prove that $\psi_1^0 \in \mathcal{A}_1^\xi$ and $\psi_2^0\in\mathcal{A}_2^\xi$, we need to prove that the bounds on the L^2 and $W^{1,2}$ norm still hold. We claim that due to the uniform bound on the gradients, we have that $\psi_1^n \to \psi_1^0$ strongly in $L^2(Q_1;\mathbb{R}^M)$, and that $\psi_2^n(y_1,\cdot)\to\psi_2^0(y_1,\cdot)$ strongly in $L^2(Q_2;\mathbb{R}^M)$ for a.e. $y_1\in Q_1$, thanks to Rellich-Kondrakov theorem.
Thanks to the weak lower semi-continuity of the norms, we then get

$$\begin{cases} \|\psi_1^0\|_{L^2(Q_1;\mathbb{R}^M)} \leq \liminf\limits_{n\to\infty}\|\psi_1^n\|_{L^2(Q_1;\mathbb{R}^M)} \leq \xi, \\ \|\nabla_{y_1}\psi_1^0\|_{L^2(Q_1;\mathbb{R}^{N\times M})} \leq \liminf\limits_{n\to\infty}\|\nabla_{y_1}\psi_1^n\|_{L^2(Q_1;\mathbb{R}^{N\times M})} \leq 1, \end{cases}$$

which proves that $\psi_1^0\in\mathcal{A}_1^\xi$. In a similar way we get

$$\begin{cases} \|\psi_2^0(y_1,\cdot)\|_{L^2(Q_2;\mathbb{R}^M)} \leq \liminf\limits_{n\to\infty}\|\psi_2^n(y_1,\cdot)\|_{L^2(Q_2;\mathbb{R}^M)} \leq \xi, \\ \|\nabla_{y_2}\psi_2^0(y_1,\cdot)\|_{L^2(Q_2;\mathbb{R}^{N\times M})} \leq \liminf\limits_{n\to\infty}\|\nabla_{y_2}\psi_2^n(y_1,\cdot)\|_{L^2(Q_2;\mathbb{R}^{N\times M})} \leq 1, \end{cases}$$

which proves that $\psi_2^0 \in \mathcal{A}_2^\xi$.

We can then conclude thanks to the boundedness assumption on W and uniform continuity of W on $\overline{B(0,0,2M)}$ (thanks to ψ_1, ψ_2 being bounded), which allows us to use Dominated Convergence Theorem:

$$\begin{aligned}\lim_{n\to\infty} \int_{Q_1}\int_{Q_2} &W(y_1, y_2, z + \psi_1^n(y_1) + \psi_2^n(y_1, y_2))\, \mathrm{d}y_1\, \mathrm{d}y_2 \\ &= \int_{Q_1}\int_{Q_2} W(y_1, y_2, z + \psi_1^0(y_1) + \psi_2^0(y_1, y_2))\, \mathrm{d}y_1\, \mathrm{d}y_2\end{aligned}$$

Step 2: Continuity of W^ξ. Take a sequence $(z_n)_n \subset \mathbb{R}^M$ such that $z_n \to z_0 \in \mathbb{R}^M$. From Step 1 we know that for every z_n there exists minimizing functions $\psi_1^n \in \mathcal{A}_1^\xi$ and $\psi_2^n \in \mathcal{A}_2^\xi$ such that

$$W^\xi(z_n) = \int_{Q_1}\int_{Q_2} W(y_1, y_2, z_n + \psi_1^n(y_1) + \psi_2^n(y_1, y_2))\, \mathrm{d}y_1\, \mathrm{d}y_2.$$

Same holds for z_0, namely there exists $\psi_1^0 \in \mathcal{A}_1^\xi$ and $\psi_2^0 \in \mathcal{A}_2^\xi$ such that

$$W^\xi(z_0) = \int_{Q_1}\int_{Q_2} W(y_1, y_2, z_0 + \psi_1^0(y_1) + \psi_2^0(y_1, y_2))\, \mathrm{d}y_1\, \mathrm{d}y_2.$$

Ussing ψ_1^n and ψ_2^n as competitors in the problem defining $W^\xi(z_0)$, we get

$$W^\xi(z_0) \le \int_{Q_1}\int_{Q_2} W(y_1, y_2, z_0 + \psi_1^n + \psi_2^n)\, \mathrm{d}y_1\, \mathrm{d}y_2. \tag{37}$$

We now want to write z_n instead of z_0 in the argument of W. We then write

$$\begin{aligned}&W(y_1, y_2, z_0 + \psi_1^n + \psi_2^n) \\ &\quad\le W(y_1, y_2, z_n + \psi_1^n + \psi_2^n) + \left|W(y_1, y_2, z_0 + \psi_1^n + \psi_2^n) - W(y_1, y_2, z_n + \psi_1^n + \psi_2^n)\right|.\end{aligned} \tag{38}$$

Since the competitors are uniformly bounded, we have

$$\lim_{n\to\infty} |W(y_1, y_2, z_0 + \psi_1^n + \psi_2^n) - W(y_1, y_2, z_n + \psi_1^n + \psi_2^n)| = 0,$$

for all $y_1 \in Q_1$ and $y_2 \in Q_2$. Then, using Dominated Convergence Theorem, we can conclude that

$$\lim_{n\to\infty} \int_{Q_1}\int_{Q_2} |W(y_1, y_2, z_0 + \psi_1^n + \psi_2^n) - W(y_1, y_2, z_n + \psi_1^n + \psi_2^n)|\, \mathrm{d}y_1\, \mathrm{d}y_2 = 0. \tag{39}$$

Therefore, from (37), (38) and (39) we obtain that

$$W^\xi(z_0) \le \liminf_{n\to\infty} \int_{Q_1}\int_{Q_2} W(y_1, y_2, z_n + \psi_1^n + \psi_2^n)\, \mathrm{d}y_1\, \mathrm{d}y_2 = \liminf_{n\to\infty} W^\xi(z_n). \tag{40}$$

For the other inequality we use ψ_1^0 and ψ_2^0 as competitors in the problem defining $W^\xi(z_n)$. This yields

$$W^\xi(z_n) \le \int_{Q_1}\int_{Q_2} W(y_1, y_2, z_n + \psi_1^0 + \psi_2^0)\, \mathrm{d}y_1\, \mathrm{d}y_2. \tag{41}$$

We now want to write z_0 instead of z_n in the argument of W. We then write

$$\begin{aligned}&W(y_1, y_2, z_n + \psi_1^0 + \psi_2^0) \\ &\quad\le W(y_1, y_2, z_0 + \psi_1^0 + \psi_2^0) + \left|W(y_1, y_2, z_n + \psi_1^0 + \psi_2^0) - W(y_1, y_2, z_0 + \psi_1^0 + \psi_2^0)\right|\end{aligned} \tag{42}$$

Since the competitors are uniformly bounded, we have

$$\lim_{n\to\infty} \left|W(y_1, y_2, z_n + \psi_1^0 + \psi_2^0) - W(y_1, y_2, z_0 + \psi_1^0 + \psi_2^0)\right| = 0,$$

for all $y_1 \in Q_1$ and $y_2 \in Q_2$. Then, using Dominated Convergence Theorem, we can conclude that

$$\lim_{n\to\infty} \int_{Q_1}\int_{Q_2} |W(y_1, y_2, z_0 + \psi_1^n + \psi_2^n) - W(y_1, y_2, z_n + \psi_1^n + \psi_2^n)| \, \mathrm{d}y_1 \, \mathrm{d}y_2 = 0. \tag{43}$$

Therefore, (41), (42) and (43) give us that

$$\limsup_{n\to\infty} W^\xi(z_n) \le \int_{Q_1}\int_{Q_2} W(y_1, y_2, z_0 + \psi_1^0 + \psi_2^0) \, \mathrm{d}y_1 \, \mathrm{d}y_2 = W^\xi(z_0). \tag{44}$$

Thus, from (40) and (44) we conclude that

$$\lim_{n\to\infty} W^\xi(z_n) = W^\xi(z_0),$$

as desired.

Step 3: Double-well behavior. Let $z \in \{a, b\}$. Take now $\psi_1 \equiv 0$, $\psi_2 \equiv 0$ as competitors in the definition of $W^\xi(z)$, to obtain that

$$W^\xi(z) \le \int_{Q_1}\int_{Q_2} W(y_1, y_2, z) \, \mathrm{d}y_1 \, \mathrm{d}y_2 = 0,$$

since $z \in \{a, b\}$ implies $W(y_1, y_2, z) = 0$ for all $y_1 \in Q_1$, $y_2 \in Q_2$. Since $W^\xi(z) \ge 0$, this implies $W^\xi(z) = 0$ if $z \in \{a, b\}$.
Let now $z \in \mathbb{R}^M$ be such that $W^\xi(z) = 0$. Using Step 1, we know that there exist minimizing functions $\psi_1 \in \mathcal{A}_1^\xi$ and $\psi_2 \in \mathcal{A}_2^\xi$ such that

$$\int_{Q_1}\int_{Q_2} W(y_1, y_2, z + \psi_1(y_1; z) + \psi_2(y_1, y_2; z)) \, \mathrm{d}y_2 \, \mathrm{d}y_1 = 0.$$

As $W(y_1, y_2, p) \ge 0$, this implies

$$W(y_1, y_2, z + \psi_1(y_1; z) + \psi_2(y_1, y_2; z)) = 0 \qquad \text{for a.e. } y_1 \in Q_1, y_2 \in Q_2.$$

Using (H2) we now know that this implies

$$z + \psi_1(y_1; z) + \psi_2(y_1, y_2; z) \in \{a, b\} \qquad \text{for a.e. } y_1 \in Q_1, y_2 \in Q_2.$$

Let's fix now also $y_1 \in Q_1$. We are left with

$$z + \psi_1(y_1; z) + \psi_2(y_1, y_2; z) \in \{a, b\} \qquad \text{for a.e. } y_2 \in Q_2,$$

which can only be true if the function $y_1 \mapsto \psi_2(y_1, \cdot; z)$ is a constant c_1 depending only on y_1 (and on z as a parameter). Keeping in mind that ψ_2 is bounded in L^2 and in L^∞, since $\psi_2 \to 0$ in L^2 as $\xi \to 0$, using Egoroff theorem we get that $\psi_2 \to 0$ uniformly on a compact $K \subset Q_1 \times Q_2$ such that $\mathcal{L}^N((Q_1 \times Q_2) \setminus K) < \varepsilon$, for an arbitrarily small $\varepsilon > 0$. This in particular implies that $c_1(y_1; z) \to 0$ uniformly on the compact given by the projection of K to Q_1.

We now have

$$z + \psi_1(y_1; z) + c_1(y_1; z) \in \{a, b\} \qquad \text{for a.e. } y_1 \in Q_1,$$

and this can only be true if $\psi_1(y_1; z) + c_1(y_1; z)$ is a constant. Due to the definition of the auxiliary problem though, it is clear that this constant must be bounded. Using now the same reasoning as before, since $\psi_1 + c_1 \to 0$ in L^2 as $\xi \to 0$, using Egoroff theorem we have that $\psi_1 + c_1 \to 0$ uniformly on a compact set $K_1 \subset Q_1$ with $\mathcal{L}^N(Q_1 \setminus K_1) < \varepsilon$. Summing everything up, for ξ sufficiently small, this implies that on an arbitrarily big compact set $K \in Q_1 \times Q_2$, we have that $\psi_1(y_1; z) + \psi_2(y_1, y_2; z)$ must be bounded by a constant $r(\xi)$ such that $r(\xi) \to 0$ as $\xi \to 0$.
Therefore we have that $z \in \overline{B(a, r(\xi))} \cup \overline{B(b, r(\xi))}$, which we can always assume disjoint for ξ small enough.

Step 4: Convergence of W^ξ as $\xi \to 0$. Fix $z \in \mathbb{R}^M$, and let $\xi_1, \xi_2 \in \mathbb{R}$ such that $\xi_1 \leq \xi_2$. it is possible to observe that

$$\mathcal{A}_1^{\xi_1} \subseteq \mathcal{A}_1^{\xi_2}, \qquad \mathcal{A}_2^{\xi_1} \subseteq \mathcal{A}_2^{\xi_2},$$

which implies

$$W^{\xi_2}(z) \leq W^{\xi_1}(z).$$

Therefore $W^\xi(z)$ is non-increasing in ξ, and since $W^\xi(z) \geq 0$, this in turn implies that the limit exists and is finite. Let's now take a sequence $(\xi_n)_n \subset \mathbb{R}$ such that $\xi_n \to 0$ as $n \to \infty$. To each ξ_n associate the respective problem $W^{\xi_n}(z)$, which, by Step 1, will have two minimizing functions $\psi_1^n \in \mathcal{A}_1^{\xi_n}$ and $\psi_2^n \in \mathcal{A}_1^{\xi_n}$, such that

$$W^{\xi_n}(z) = \int_{Q_1} \int_{Q_2} W(y_1, y_2, z + \psi_1^n + \psi_2^n) \,\mathrm{d}y_1 \,\mathrm{d}y_2.$$

We now want to study the behaviour of $W^{\xi_n}(z)$ as $\xi_n \to 0$. We know already that

$$\|\psi_1^n\|_{L^2(Q_1;\mathbb{R}^M)} \leq \xi, \qquad \|\psi_2^n\|_{L^2(Q_1;L^2(Q_2;\mathbb{R}^M))} \leq \xi.$$

This therefore implies that

$$\psi_1^n \to 0 \quad \text{in } L^2(Q_1;\mathbb{R}^M), \qquad \psi_2^n \to 0 \quad \text{in } L^2(Q_1; L^2(Q_2;\mathbb{R}^M)).$$

Then again, using the uniform bound on the competitors, the boundedness assumption on W and Dominated Convergence Theorem, we get that

$$\lim_{n\to\infty} W^{\xi_n}(z) = W^{\mathrm{h}}(z),$$

where $W^{\mathrm{h}}(z)$ is given by

$$W^{\mathrm{h}}(z) = \int_{Q_1} \int_{Q_2} W(y_1, y_2, z) \,\mathrm{d}y_1 \,\mathrm{d}y_2.$$

Since the convergence is non-increasing, we can use Dini's Theorem to deduce that the convergence is uniform on compact sets. $\square$

4.3. **Limit of the auxiliary surface tension.** The goal of this section is to prove that

$$\lim_{\xi\to 0} \sigma^\xi = \sigma^{\mathrm{h}},$$

where

$$\sigma^\xi := \inf \left\{ \int_{-1}^{1} 2\sqrt{W^\xi(\gamma(t))}|\gamma'(t)| \,\mathrm{d}t : \gamma \in \mathrm{Lip}([-1,1];\mathbb{R}^M), \gamma(-1) = a, \gamma(1) = b \right\}. \tag{45}$$

First of all, we show that

$$\sigma^\xi = \min \left\{ \int_{-1}^{1} 2\sqrt{W^\xi(\gamma(t))}|\gamma'(t)| \,\mathrm{d}t : \gamma \in \mathrm{Lip}_{\mathcal{Z}^\xi}([-1,1];\mathbb{R}^M), \gamma(-1) = a, \gamma(1) = b \right\},$$

where $\mathrm{Lip}_{\mathcal{Z}^\xi}([-1,1];\mathbb{R}^M)$ is the space of continuous curves which are Lipschitz continuous with respect to the Euclidean metric on any compact portion of the curve that does not intersect

$$\mathcal{Z}^\xi := \overline{B(a, r(\xi))} \cup \overline{B(b, r(\xi))}.$$

Here, $r(\xi) > 0$ is given by Proposition 34.

Remark 35. Recall that

$$\lim_{\xi\to 0} r(\xi) = 0. \tag{46}$$

In particular, we can always assume that ξ is small enough so that $\overline{B(a, r(\xi))}$ and $\overline{B(b, r(\xi))}$ are disjoint.

The strategy of the proof is similar to that employed to prove [41, Theorem 2.6]. Therefore, we only highlight the main change. Consider the functional $E^\xi : \mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M) \to [0,\infty)$ defined as

$$E^\xi(\gamma) := \int_{-1}^{1} \sqrt{W^\xi(\gamma(t))}|\gamma'(t)|\,\mathrm{d}t.$$

for $p,q \in \mathbb{R}^M$, set

$$\mathrm{d}^\xi(p,q) := \inf\left\{\int_{-1}^{1} E^\xi(\gamma(t))|\gamma'(t)|\,\mathrm{d}t : \gamma \in \mathrm{Lip}_{\mathcal{Z}^\xi}([-1,1];\mathbb{R}^M), \gamma(-1)=p, \gamma(1)=q\right\}.$$

Remark 36. Note that the function $\mathrm{d} : \mathbb{R}^M \times \mathbb{R}^M \to [0,\infty)$ defined above turns out to be a quasi-metric. Indeed, if $p,q \in B(a,r(\xi))$ or $p,q \in B(b,r(\xi))$, then $\mathrm{d}(p,q)=0$ does not imply that $p=q$. Nevertheless, it is possible to see that d is a metric on $\mathbb{R}^M/\sim$, where $p \sim q$ if $p,q \in B(a,r(\xi))$ or $p,q \in B(b,r(\xi))$. In this way we get that $(\mathbb{R}^M/\sim, \mathrm{d})$ is a length space.

Let $\xi > 0$, and fix $p,q \in \mathbb{R}^M$ such that $p \in \overline{B(a,\xi)}$ and $q \in \overline{B(b,\xi)}$. Take a curve $\gamma \in \mathrm{Lip}_{\mathcal{Z}^\xi}([-1,1];\mathbb{R}^M)$ such that $\gamma(-1)=p$ and $\gamma(1)=q$. In light of the minimization problem defining $\mathrm{d}^\xi(p,q)$, without loss of generality, we can assume that there exist $-1 \le t_a^\xi < t_b^\xi \le 1$ such that

$$\gamma(t) \in \overline{B(a,r(\xi))} \quad \text{for all } t \in [-1,t_a^\xi], \qquad \gamma(t) \in \overline{B(b,r(\xi))} \quad \text{for all } t \in [t_b^\xi,1].$$

We observe that

$$\lim_{\xi\to 0} t_a^\xi \to -1, \qquad \lim_{\xi\to 0} t_b^\xi \to 1.$$

Define $\widetilde{\gamma} \in \mathrm{Lip}_{\mathcal{Z}^\xi}([-1,1];\mathbb{R}^M)$ as

$$\widetilde{\gamma}(t) := \begin{cases} p + \frac{3}{2}(\gamma(t_a^\xi)-p)(t+1) & t \in \left[-1,-\frac{1}{3}\right), \\ \gamma\left(t_a^\xi + \frac{t_b^\xi - t_a^\xi}{2}(3t+1)\right) & t \in \left[-\frac{1}{3},\frac{1}{3}\right], \\ q - \frac{3}{2}(\gamma(t_b^\xi)-q)(t-1) & t \in \left[\frac{1}{3},1\right). \end{cases}$$

Note that $\widetilde{\gamma}(-1)=p$ and $\widetilde{\gamma}(1)=q$, and that

$$\begin{aligned} E(\widetilde{\gamma}) &= \int_{-1}^{1} F(\widetilde{\gamma}(t))|\widetilde{\gamma}'(t)|\,\mathrm{d}t = \int_{-\frac{1}{3}}^{\frac{1}{3}} F(\widetilde{\gamma}(t))|\widetilde{\gamma}'(t)|\,\mathrm{d}t \\ &= \int_{-\frac{1}{3}}^{\frac{1}{3}} F\left(\gamma\left(t_a^\xi + \frac{t_b^\xi - t_a^\xi}{2}(3t+1)\right)\right)\left|\gamma'\left(t_a^\xi + \frac{t_b^\xi - t_a^\xi}{2}(3t+1)\right)\right|\frac{3}{2}\left|t_b^\xi - t_a^\xi\right|\,\mathrm{d}t \\ &= \int_{t_a^\xi}^{t_b^\xi} F(\gamma(s))|\gamma'(s)|\,\mathrm{d}s = \int_{-1}^{1} F(\gamma(s))|\gamma'(s)|\,\mathrm{d}s = E(\gamma) \end{aligned}$$

Therefore, in the following, we will always assume that

$$\gamma(t) \in \overline{B(a,r(\xi))} \quad \text{for all } t \in \left[-1,-\frac{1}{3}\right], \qquad \gamma(t) \in \overline{B(b,r(\xi))} \quad \text{for all } t \in \left[\frac{1}{3},1\right]. \tag{47}$$

Define the length functional as

$$L(\gamma) := \sup_{(t_k)_k \subset \mathcal{P}} \sum_k \mathrm{d}(\widetilde{\gamma}(t_k),\widetilde{\gamma}(t_{k+1})),$$

where $\mathcal{P}$ is the set of finite partitions of $\left[-\frac{1}{3},\frac{1}{3}\right]$. With these definitions, the proof of [41, Theorem 2.6] yields the following.

Proposition 37. *Fix $\xi > 0$. For every points $p, q \in \mathbb{R}^M$ there exists a curve $\gamma^* \in \mathrm{Lip}_{\mathcal{Z}^\xi}([-1,1];\mathbb{R}^M)$ such that*
$$\mathrm{d}(p,q) = E(\gamma^*) = L(\gamma^*),$$
where we assume that the curve is parametrized to satisfy (47).

Now, for $p, q \in \mathbb{R}^M$, define
$$\mathrm{d}^0(p,q) := \lim_{\xi \to 0} \mathrm{d}^\xi(p,q).$$
Note that the limit exists, since the function W^ξ in increasing (see Proposition 34). Using the same strategy as in [17, Lemma 3.5], we get

Lemma 38. *The function $\mathrm{d}^0 : \mathbb{R}^M \times \mathbb{R}^M \to [0,\infty)$ is a metric on $\mathbb{R}^M$.*

We now need a technical result whose proof follows the same lines as those of the proofs of [17, Lemma 3.6, Proposition 3.7]. Indeed, since we only restrict ourselves to the region where the curve lies outside of its zeros, we are in the same exact assumptions as Moreover, note that by (46), the zeros of W^ξ collapses to the points a and b, as $\xi \to 0$.

Proposition 39. *Let $(\xi_n)_n$ be an infinitesimal sequence, and for each $n \in \mathbb{N}$, let γ_{ξ_n} be a geodesic of $\mathrm{d}^{\xi_n}(a,b)$. Then, up to a subsequence, there exists $\gamma_0 \in \mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M)$ with $\gamma(-1) = a$, $\gamma(1) = b$ such that*
$$\lim_{n\to\infty} \sup_{t \in [-\frac{1}{3},\frac{1}{3}]} \mathrm{d}^0(\widetilde{\gamma}_{\xi_n}(t), \widetilde{\gamma}_0(t)) = 0,$$
Moreover,
$$\mathrm{d}^0(a,b) = \lim_{\xi\to 0} \int_{-1}^{1} 2\sqrt{W^\xi(\gamma_0)}|\gamma_0'| \,\mathrm{d}t.$$

Remark 40. Note that the above result holds for every pair of points $p, q \in \mathbb{R}^M$ as end points in place of a and b, respectively.

We are now in position to prove the main result of this section.

Proposition 41. *It holds that*
$$\lim_{\xi\to 0} \sigma^\xi = \sigma^{\mathrm{h}},$$
where σ^ξ and σ^{h} are defined in (45) and (16), respectively.

Proof. **Step 1: we show that $\sigma^0 \leq \sigma^{\mathrm{h}}$.** Since W^ξ is increasingly converging to W^{h} as $\xi \to 0$ (see Proposition 34), this follows from the definition of σ^0.

Step 2: we show that $\sigma^0 \geq \sigma^{\mathrm{h}}$. The proof follows the same lines as that of [17, Proposition 4.6]. We report it in here for the reader's convenience. First of all, note that since
$$\lim_{\xi\to 0} \sigma^\xi = \sup_{\xi > 0} \sigma^\xi,$$
that we get the desired result if we prove that there exists an infinitesimal sequence $(\xi_n)_n$ such that
$$\lim_{n\to\infty} \sigma^{\xi_n} \geq \sigma^{\mathrm{h}}.$$
Let $\gamma_0 \in \mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M)$ be the curve given by Proposition 39. For $n \in \mathbb{N} \setminus \{0\}$, define
$$T_n^a := \left\{ t \in [-1,1] : \gamma_0(t) \notin \overline{B\left(a, r(\xi_n) + \frac{1}{n}\right)} \right\},$$
$$T_n^b := \left\{ t \in [-1,1] : \gamma_0(t) \notin \overline{B\left(b, r(\xi_n) + \frac{1}{n}\right)} \right\}.$$

Note that, by assumption (47), we have that

$$T_n^a = [-1, t_n^a], \qquad T_n^a = [t_n^b, 1],$$

where

$$\lim_{n\to\infty} t_n^a = -1, \qquad \lim_{n\to\infty} t_n^b = 1. \tag{48}$$

Moreover, note that, since γ_0 is Lipschitz in $[t_n^a, t_n^b]$, it holds that

$$L_n := \int_{t_n^a}^{t_n^b} |\gamma_0'| \,\mathrm{d}t < \infty,$$

for all $n \in \mathbb{N} \setminus \{0\}$. Let $R > 0$ be such that $\gamma_0(t) \in B(0, R)$ for all $t \in [-1, 1]$. Using the uniform convergence of W^{ξ_n} to W^{h}, we can find an infinitesimal sequence $(\xi_n)_n$ such that

$$\|\sqrt{W^{\xi_n}} - \sqrt{W^{\mathrm{h}}}\|_{C^0(B(0,R))} < \frac{1}{2nL_n}, \tag{49}$$

for all $n \in \mathbb{N} \setminus \{0\}$. Thus, for each $n \in \mathbb{N}$, we get

$$\begin{aligned}
\sigma^0 &= \sup_{\xi>0} \int_{-1}^{1} 2\sqrt{W^{\xi}(\gamma_0)}|\gamma_0'| \,\mathrm{d}t \geq \int_{-1}^{1} 2\sqrt{W^{\xi_n}(\gamma_0)}|\gamma_0'| \,\mathrm{d}t \geq \int_{t_n^a}^{t_n^b} 2\sqrt{W^{\xi_n}(\gamma_0)}|\gamma_0'| \,\mathrm{d}t \\
&\geq \int_{t_n^a}^{t_n^b} 2\sqrt{W^{\mathrm{h}}(\gamma_0)}|\gamma_0'| \,\mathrm{d}t - 2\int_{t_n^a}^{t_n^b} \left|\sqrt{W^{\xi_n}(\gamma_0)} - \sqrt{W^{\mathrm{h}}(\gamma_0)}\right| |\gamma_0'| \,\mathrm{d}t \\
&\geq \int_{t_n^a}^{t_n^b} 2\sqrt{W^{\mathrm{h}}(\gamma_0)}|\gamma_0'| \,\mathrm{d}t - \frac{1}{n},
\end{aligned}$$

where in the last step we used (49). Taking the limit as $n \to \infty$ on both sides, and using the Dominated Convergence Theorem, yields the desired result. □

5. Compactness

In this section we prove Theorem 6.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset W^{1,2}(\Omega; \mathbb{R}^M)$ be a sequence such that

$$\sup_{n\in\mathbb{N}} F_n^{(1)}(u_n) \leq C < +\infty.$$

Using (H3), we have

$$\sup_{n\in\mathbb{N}} \int_{\Omega} \left[\frac{1}{\varepsilon_n} W_1(u_n) + \varepsilon_n |\nabla u_n|^2\right] \mathrm{d}x \leq \sup_{n\in\mathbb{N}} F_n^{(1)}(u_n) \leq C < +\infty.$$

This allows us to use the methods of [26] in order to extract a subsequence $(u_{n_k})_{k\in\mathbb{N}} \subset W^{1,2}(\Omega; \mathbb{R}^M)$ such that $u_{n_k} \to u \in \mathrm{BV}(\Omega; \{a, b\})$ strongly in $L^1(\Omega; \mathbb{R}^M)$. □

6. Liminf inequality

The goal of this section is to prove the following proposition.

Proposition 42. *Let $(u_n)_{n\in\mathbb{N}} \subset W^{1,2}(\Omega; \mathbb{R}^M)$ be a sequence such that $u_n \to u \in \mathrm{BV}(\Omega; \{a, b\})$ strongly in $L^1(\Omega; \mathbb{R}^M)$. Then, it holds that*

$$\liminf_{n\to\infty} F_n^{(1)}(u_n) \geq F_\infty^{(1)}(u).$$

Proof. **Step 1: Reduction to uniformly bounded u_n and W linear outside of a ball.** Let $M>0$ such that $M>R$, where $R>0$ is the constant appearing in (H3). Let $\varphi_M\colon[0,+\infty)\to[0,1]$ be a smooth cut-off function such that

$$\varphi_M(t)\begin{cases}=1 & t\in[0,M],\\ \in(0,1) & t\in(M,2M),\\ =0 & t\in[2M,+\infty).\end{cases}$$

We now modify W such that it is linear outside of a ball of radius $2M$: to do that, we define $\widetilde{W}_M\colon\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^M\to[0,+\infty)$ as

$$\widetilde{W}_M(y_1,y_2,z):=\varphi_M(|z|)W(y_1,y_2,z)+(1-\varphi_M(|z|))\frac{|z|}{R}.$$

Observe that this function still satisfies all the needed assumptions, that is (H1), (H2), (H3) and (H4). In this way, we have

$$\widetilde{W}_M(y_1,y_2,z)\begin{cases}=W(y_1,y_2,z) & |z|\le M,\\ \ge\frac{|z|}{R} & |z|\ge R,\\ =\frac{|z|}{R} & |z|\ge 2M.\end{cases}$$

We can then define

$$\widetilde{F}^{(1)}_{n,M}(v):=\int_\Omega\left[\frac{1}{\varepsilon_n}\widetilde{W}_M\left(\frac{x}{\delta_n},\frac{x}{\eta_n},v\right)+\varepsilon_n|\nabla v|^2\right]\mathrm{d}x.$$

Our goal is to now prove that, if

$$F^{(1)}_\infty(u)\le\liminf_{n\to\infty}\widetilde{F}^{(1)}_{n,M}(v_n),\tag{50}$$

holds for all $u\in\mathrm{BV}(\Omega;\{a,b\})$ and all sequences $(v_n)_n\subset W^{1,2}(\Omega;\mathbb{R}^M)$ such that $v_n\to u$ in $L^1(\Omega;\mathbb{R}^M)$ and $\|v_n\|_\infty\le 2M$, then

$$F^{(1)}_\infty(u)\le\liminf_{n\to\infty}F^{(1)}_n(u_n),\tag{51}$$

holds for all sequences $(u_n)_n\subset W^{1,2}(\Omega;\mathbb{R}^M)$ such that $u_n\to u$ in $L^1(\Omega;\mathbb{R}^M)$.
Let us fix $u\in\mathrm{BV}(\Omega;\{a,b\})$ and a sequence $(u_n)_n\subset W^{1,2}(\Omega;\mathbb{R}^M)$ with $u_n\to u$ in $L^1(\Omega;\mathbb{R}^M)$. Using the truncation operator defined in (36) we define

$$v_n:=\mathcal{T}_{2M}u_n=\begin{cases}u_n & |u_n|\le 2M,\\ 2M\frac{u_n}{|u_n|} & |u_n|>2M.\end{cases}$$

It is an easy consequence of this definition that

$$|\nabla v_n|=|\nabla\mathcal{T}_{2M}u_n|\le|\nabla u_n|.\tag{52}$$

Observe that $\|v_n\|_\infty\le 2M$ for every $n\in\mathbb{N}$. Therefore, (50) holds if we prove that $v_n\to u$ in $L^1(\Omega;\mathbb{R}^M)$. Using the triangle inequality it is enough to prove that $v_n-u_n\to 0$ in $L^1(\Omega;\mathbb{R}^M)$, as

$$\|v_n-u\|_{L^1(\Omega;\mathbb{R}^M)}\le\|v_n-u_n\|_{L^1(\Omega;\mathbb{R}^M)}+\|u_n-u\|_{L^1(\Omega;\mathbb{R}^M)}.$$

The definition of truncation operator yields that:

$$\begin{aligned}\|u_n-v_n\|_{L^1(\Omega;\mathbb{R}^M)}&=\int_\Omega|u_n-v_n|\,\mathrm{d}x\\ &=\int_{\{|u_n|\le 2M\}}|u_n-u_n|\,\mathrm{d}x+\int_{\{|u_n|>2M\}}|u_n-v_n|\,\mathrm{d}x\\ &=\int_{\{|u_n|>2M\}}|u_n-v_n|\,\mathrm{d}x.\end{aligned}\tag{53}$$

Now let $\Lambda_n \coloneqq \{x \in \Omega : |u_n(x)| > 2M\}$. Using Chebychev's inequality and Assumption (H3) we have

$$|\Lambda_n| \leq \frac{1}{2M}\int_{\Lambda_n}|u_n|\,\mathrm{d}x \leq \frac{1}{2M}\int_{\Omega}|u_n|\,\mathrm{d}x \leq \frac{R}{2M}\int_{\Omega}W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n\right)\,\mathrm{d}x \leq c\frac{R}{2M}\varepsilon_n.$$

Going back to (53), using triangle inequality and Assumption (H3) we now have

$$\begin{aligned}\|u_n - v_n\|_{L^1(\Omega;\mathbb{R}^M)} &= \int_{\Lambda_n}|u_n - v_n|\,\mathrm{d}x\\ &\leq \int_{\Lambda_n}|u_n|\,\mathrm{d}x + \int_{\Lambda_n}|v_n|\,\mathrm{d}x\\ &\leq R\int_{\Lambda_n}W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n\right)\,\mathrm{d}x + \int_{\Lambda_n}2M\,\mathrm{d}x\\ &\leq R\int_{\Omega}W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n\right)\,\mathrm{d}x + 2M|\Lambda_n|\\ &\leq cR\varepsilon_n + cR\varepsilon_n = C\varepsilon_n,\end{aligned}$$

and therefore we conclude.
In order to prove (51) we need to prove that $\widetilde{F}^{(1)}_{n,M}(v_n) \leq F^{(1)}_n(u_n)$. This, together with (50), will let us conclude.
From the definition of Λ_n and the truncation operator we have that for $x \in \Lambda_n$ it holds

$$|u_n(x)| > 2M = |\mathcal{T}_{2M}u_n(x)| = |v_n(x)|.$$

Therefore, for $x \in \Lambda_n$ we have

$$\widetilde{W}_M\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n(x)\right) = \frac{|u_n(x)|}{R} \geq \frac{|v_n(x)|}{R} = \widetilde{W}_M\left(\frac{x}{\delta_n},\frac{x}{\eta_n},v_n(x)\right).$$

For $x \in \Omega \setminus \Lambda_n$ it holds that $v_n(x) = \mathcal{T}_{2M}u_n(x) = u_n(x)$, therefore

$$\widetilde{W}_M\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n(x)\right) = \widetilde{W}_M\left(\frac{x}{\delta_n},\frac{x}{\eta_n},v_n(x)\right).$$

This implies that

$$\widetilde{W}_M\left(\frac{x}{\delta_n},\frac{x}{\eta_n},v_n(x)\right) \leq \widetilde{W}_M\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n(x)\right) \leq W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n(x)\right),$$

which, together with (52) allows us to conclude.
Therefore from now on we assume that W has linear growth at infinity and that $\|u_n\|_\infty \leq 2M$.

Step 2: Let $(u_n)_n \subset W^{1,2}(\Omega;\mathbb{R}^M)$ be a sequence converging to $u \in \mathrm{BV}(\Omega;\{a,b\})$ in $L^1(\Omega;\mathbb{R}^M)$, such that $\|u_n\|_\infty \leq 2M$. We claim that

$$\liminf_{n\to\infty} F^{(1)}_n(u_n) \geq \liminf_{n\to\infty}\int_\Omega 2\sqrt{W^\xi(u_n)}|\nabla u_n|\,\mathrm{d}x. \tag{54}$$

Let us define now

$$\begin{aligned}\widetilde{\Omega}_n &\coloneqq \{x \in \Omega : \|\nabla_{y_1}\mathcal{U}_1 u_n(x,\cdot)\|_{L^2(Q_1;\mathbb{R}^{N\times M})} \leq 1\},\\ \widetilde{Q}_{1,n}(x) &\coloneqq \{y_1 \in Q_1 : \|\nabla_{y_2}\mathcal{U}_2 u_n(x,y_1,\cdot)\|_{L^2(Q_2;R^{N\times M})} \leq 1\}.\end{aligned}$$

Using Chebychev's inequality we have

$$|\Omega \setminus \widetilde{\Omega}_n| \leq \int_\Omega \|\nabla_{y_1}\mathcal{U}_1 u_n(x,\cdot)\|^2_{L^2(Q_1;\mathbb{R}^{N\times M})}\,\mathrm{d}x = \|\nabla_{y_1}\mathcal{U}_1 u_n\|^2_{L^2(\Omega;L^2(Q_1;\mathbb{R}^{N\times M}))} \leq c\frac{\delta_n^2}{\varepsilon_n} \to 0.$$

For $\widetilde{Q}_{1,n}(x)$, using again Chebychev's inequality we have that

$$|Q_1 \setminus \widetilde{Q}_{1,n}(x)| \leq \int_{Q_1}\|\nabla_{y_2}\mathcal{U}_2 u_n(x,y_1,\cdot)\|^2_{L^2(Q_2;\mathbb{R}^{N\times M})}\,\mathrm{d}y_1 = \|\nabla_{y_2}\mathcal{U}_2 u_n(x,\cdot,\cdot)\|^2_{L^2(Q_1\times Q_2;\mathbb{R}^{N\times M})}.$$

Note that

$$|Q_1 \setminus \widetilde{Q}_{1,n}(x)| \leq |Q_1| = 1,$$

and that

$$\begin{aligned}\int_\Omega |Q_1 \setminus \widetilde{Q}_{1,n}(x)| \,\mathrm{d}x &\leq \int_\Omega \|\nabla_{y_2}\mathcal{U}_2 u_n(x,\cdot,\cdot)\|^2_{L^2(Q_1\times Q_2;\mathbb{R}^{N\times M})} \,\mathrm{d}x \\ &= \|\nabla_{y_2}\mathcal{U}_2 u_n\|^2_{L^2(\Omega;L^2(Q_1\times Q_2;\mathbb{R}^{N\times M}))} \leq c\frac{\eta_n^2}{\varepsilon_n} \to 0,\end{aligned}$$

from which we can conclude that

$$|Q_1 \setminus \widetilde{Q}_{1,n}(x)| \to 0 \qquad \text{for a.e. } x \in \Omega.$$

Let us also define a variation of the auxiliary problem. Take $z \in \mathbb{R}^M$ and $K \subset Q_1$ a compact set. Then we define $W^\xi(z, K)$ as

$$W^\xi(z,K) := \inf_{\psi_1\in\mathcal{A}_1^\xi} \inf_{\psi_2\in\mathcal{A}_2^\xi} \int_K \int_{Q_2} W(y_1, y_2, z + \psi_1(y_1) + \psi_2(y_1, y_2)) \,\mathrm{d}y_2 \,\mathrm{d}y_1,$$

where $\mathcal{A}_1^\xi$ and $\mathcal{A}_2^\xi$ are defined as in Definition 33. Note that if $K = Q_1$, then $W^\xi(z, Q_1) = W^\xi(z)$.

Thanks to the non-negativity of W, together with Lemma 19, we have

$$\int_\Omega W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n(x)\right) \,\mathrm{d}x \geq \int_{\widetilde{\Omega}_n} \int_{\widetilde{Q}_{1,n}(x)} \int_{Q_2} W(y_1, y_2, \mathcal{U}_2 u_n) \,\mathrm{d}y_2 \,\mathrm{d}y_1 \,\mathrm{d}x.$$

We rewrite this last equation as

$$\int_{\widetilde{\Omega}_n} \int_{\widetilde{Q}_{1,n}(x)} \int_{Q_2} W(y_1, y_2, u_n + [\mathcal{U}_1 u_n - u_n] + [\mathcal{U}_2 u_n - \mathcal{U}_1 u_n]) \,\mathrm{d}y_1 \,\mathrm{d}y_2 \,\mathrm{d}x.$$

Let us define the following functions:

$$\psi_1(y_1; x) := \mathcal{U}_1 u_n(x, y_1) - u_n(x), \qquad \psi_2(y_1, y_2; x) := \mathcal{U}_2 u_n(x, y_1, y_2) - \mathcal{U}_1 u_n(x, y_1).$$

Thanks to Theorem 32, we know that

$$\|\psi_1\|_{L^2(\Omega;L^2(Q_1;\mathbb{R}^M))} \leq C\sqrt{\delta_n}, \qquad \|\psi_2\|_{L^2(\Omega;L^2(Q_1;L^2(Q_2;\mathbb{R}^M)))} \leq C\sqrt{\frac{\eta_n}{\delta_n}}, \tag{55}$$

and since $\|\psi_1\|_\infty \leq 2M$ and $\|\psi_2\|_\infty \leq 2M$, we also have that

$$\|\psi_1\|_{L^2(Q_1;\mathbb{R}^M)} \leq C\sqrt{\delta_n}, \qquad \|\psi_2\|_{L^2(Q_1;L^2(Q_2;\mathbb{R}^M))} \leq C\sqrt{\frac{\eta_n}{\delta_n}}. \tag{56}$$

Thanks to the definitions of $\widetilde{\Omega}_n$ and $\widetilde{Q}_{1,n}(x)$, we also know that

$$\|\nabla_{y_1}\psi_1(x,\cdot)\|_{L^2(Q_1;\mathbb{R}^{N\times M})} \leq 1, \qquad \|\nabla_{y_2}\psi_2(x, y_1, \cdot)\|_{L^2(Q_2;\mathbb{R}^{N\times M})} \leq 1.$$

We now need to prove that $\psi_1 \in \mathcal{A}_1^\xi$ and $\psi_2 \in \mathcal{A}_2^\xi$. Fix $\xi > 0$. Then, take n large enough such that

$$\max\left\{\sqrt{\delta_n}, \sqrt{\frac{\eta_n}{\delta_n}}\right\} \leq \xi.$$

This is possible since all these quantities are infinitesimal for $n \to \infty$. Thus, (55) and (56) yield that ψ_1 and ψ_2 are admissible functions for the problem defining $W^\xi(u_n(x), \widetilde{Q}_{1,n}(x))$.

We can therefore write

$$\begin{aligned}\int_{\widetilde{\Omega}_n} \int_{\widetilde{Q}_{1,n}(x)} \int_{Q_2} W(y_1, y_2, \mathcal{U}_2 u_n) \,\mathrm{d}y_2 \,\mathrm{d}y_1 \,\mathrm{d}x &\geq \int_{\widetilde{\Omega}_n} W^\xi(u_n(x), \widetilde{Q}_{1,n}(x)) \,\mathrm{d}x \\ &= \int_\Omega W^\xi(u_n(x), \widetilde{Q}_{1,n}(x)) \,\mathrm{d}x - \int_{\Omega\setminus\widetilde{\Omega}_n} W^\xi(u_n(x), \widetilde{Q}_{1,n}(x)) \,\mathrm{d}x\end{aligned}$$

We claim now that

$$\lim_{n\to\infty} \int_{\Omega\setminus\widetilde{\Omega}_n} W^\xi(u_n(x), \widetilde{Q}_{1,n}(x)) \,\mathrm{d}x = 0.$$

This is easy to see, since by non-negativity of W and Assumption (H4) we get that

$$\int_{\Omega\setminus\widetilde{\Omega}_n} W^\xi(u_n(x),\widetilde{Q}_{1,n}(x))\,\mathrm{d}x \leq \int_{\Omega\setminus\widetilde{\Omega}_n} W^\xi(u_n(x))\,\mathrm{d}x \leq C_{2M}|\Omega\setminus\widetilde{\Omega}_n| \to 0.$$

Therefore we are left with

$$\liminf_{n\to\infty}\int_\Omega W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n(x)\right)\,\mathrm{d}x \geq \liminf_{n\to\infty}\int_\Omega W^\xi(u_n(x),\widetilde{Q}_{1,n}(x))\,\mathrm{d}x.$$

Take now ψ_1 and ψ_2 as minimizing functions for the problem defining $W^\xi(u_n(x),\widetilde{Q}_{1,n}(x))$, and define

$$\widetilde{\psi}_2(x,y_1,y_2) := \begin{cases} 0 & y_1\in Q_1\setminus\widetilde{Q}_{1,n}(x),\\ \psi_2(x,y_1,y_2) & y_1\in\widetilde{Q}_{1,n}(x).\end{cases}$$

Note that ψ_1 and $\widetilde{\psi}_2$ are admissible competitors for the problem defining $W^\xi(u_n(x),Q_1)$.

Using these modified competitors, we have

$$\begin{aligned}
&\int_\Omega W^\xi(u_n(x),\widetilde{Q}_{1,n}(x))\,\mathrm{d}x\\
&\quad=\int_\Omega\int_{\widetilde{Q}_{1,n}(x)}\int_{Q_2} W(y_1,y_2,u_n(x)+\psi_1(x,y_1)+\psi_2(x,y_1,y_2))\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x\\
&\quad=\int_\Omega\int_{Q_1}\int_{Q_2} W(y_1,y_2,u_n(x)+\psi_1(x,y_1)+\widetilde{\psi}_2(x,y_1,y_2))\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x\\
&\qquad\qquad-\int_\Omega\int_{Q_1\setminus\widetilde{Q}_{1,n}(x)}\int_{Q_2} W(y_1,y_2,u_n(x)+\psi_1(x,y_1))\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x.
\end{aligned}$$

Our claim is that

$$\lim_{n\to\infty}\int_\Omega\int_{Q_1\setminus\widetilde{Q}_{1,n}(x)}\int_{Q_2} W(y_1,y_2,u_n(x)+\psi_1(x,y_1))\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x = 0.$$

This is straightforward using Assumption (H4) and the fact that $\|\psi_1\|_\infty \leq M - \frac{|u_n(x)|}{2}$, which imply

$$\int_\Omega\int_{Q_1\setminus\widetilde{Q}_{1,n}(x)}\int_{Q_2} W(y_1,y_2,u_n(x)+\psi_1(x,y_1))\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x \leq C_{2M}|Q_1\setminus\widetilde{Q}_{1,n}(x)| \to 0.$$

Therefore, since ψ_1 and $\widetilde{\psi}_2$ are admissible competitors, we can further lower the energy by

$$\liminf_{n\to\infty}\int_\Omega W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n(x)\right)\,\mathrm{d}x \geq \liminf_{n\to\infty}\int_\Omega W^\xi(u_n(x),\widetilde{Q}_{1,n}(x))\,\mathrm{d}x \geq \liminf_{n\to\infty}\int_\Omega W^\xi(u_n(x))\,\mathrm{d}x.$$

Using this last inequality, together with Young's inequality, we get

$$\begin{aligned}
\liminf_{n\to\infty} F_n^{(1)}(u_n) &= \liminf_{n\to\infty}\int_\Omega\left[\frac{1}{\varepsilon_n}W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n(x)\right)+\varepsilon_n|\nabla u_n(x)|^2\right]\,\mathrm{d}x\\
&\geq \liminf_{n\to\infty}\int_\Omega\left[\frac{1}{\varepsilon_n}W^\xi(u_n(x))+\varepsilon_n|\nabla u_n(x)|^2\right]\,\mathrm{d}x\\
&\geq \liminf_{n\to\infty}\int_\Omega 2\sqrt{W^\xi(u_n(x))}|\nabla u_n(x)|\,\mathrm{d}x,
\end{aligned}$$

which was our initial claim.

Step 3: We now want to prove that

$$\liminf_{n\to\infty}\int_\Omega 2\sqrt{W^\xi(u_n))}|\nabla u_n|\,\mathrm{d}x \geq \sigma^\xi\,\mathrm{Per}(\{u=a\};\Omega), \tag{57}$$

where σ^ξ is defined in (45).

It is a well-known fact that in the case of a classic double-well potential, the functional in (54) is bounded below by the perimeter functional . We briefly recall now the proof in [26, Theorem

3.4], to show that nothing changes in the case of compact wells (see (15) and (16)).
By compactness, we know that $u_n \to u_0 \in \mathrm{BV}(\Omega;\{a,b\})$ in $L^1(\Omega;\mathbb{R}^M)$. Let us define the auxiliary function $\varphi^\xi\colon \mathbb{R}^M \to \mathbb{R}$ as

$$\varphi^\xi(z) := \inf\left\{\int_{-1}^1 2\sqrt{W^\xi(\gamma(t))}|\gamma'(t)|\,\mathrm{d}t : \gamma \in \mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M), \gamma(-1)=a, \gamma(1)=z\right\}.$$

We prove that this function is Lipschitz continuous. Take $z_1, z_2 \in \mathbb{R}^M$, and take $\gamma \in \mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M)$ such that $\gamma(-1)=a$ and $\gamma(1)=z_1$. With this curve we create another curve, which is going to be an admissible competitor in the infimum problem defining $\varphi^\xi(z_2)$, that is $\gamma_0 \in \mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M)$ such that $\gamma(-1)=a$ and $\gamma(1)=z_2$, which is

$$\gamma_0(t) := \begin{cases} \gamma(2t+1) & t\in[-1,0],\\ z_1 + t(z_2-z_1) & t \in (0,1]. \end{cases}$$

Therefore we get that

$$\begin{aligned}\varphi^\xi(z_2) &\leq \int_{-1}^1 2\sqrt{W^\xi(\gamma_0(t))}|\gamma_0'(t)|\,\mathrm{d}t\\ &= \int_{-1}^0 2\sqrt{W^\xi(\gamma(2t+1))}|2\gamma'(2t+1)|\,\mathrm{d}t + \int_0^1 \sqrt{W^\xi(z_1+t(z_2-z_1))}|z_2-z_1|\,\mathrm{d}t\\ &\leq \int_{-1}^1 2\sqrt{W^\xi(\gamma(t))}|\gamma'(t)|\,\mathrm{d}t + \left(\sup_{t\in(0,1)} 2\sqrt{W^\xi(z_1+t(z_2-z_1))}\right)|z_2-z_1|.\end{aligned}$$

Thanks to assumption (H4), taking the infimum over all $\gamma \in \mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M)$ such that $\gamma(-1)=a$ and $\gamma(1)=z_1$, we get

$$\varphi^\xi(z_2) \leq \varphi^\xi(z_1) + C|z_2-z_1|.$$

Writing the same procedure with z_1 and z_2 swapped we find

$$|\varphi^\xi(z_2)-\varphi^\xi(z_1)| \leq C|z_2-z_1|,$$

therefore proving that φ^ξ is Lipschitz continuous. Rademacher's theorem then implies that it is differentiable a.e. in $\mathbb{R}^M$. Take a differentiability point $z_0\in\mathbb{R}^M$. Using the previous computations, we get

$$\lim_{z\to z_0}\frac{|\varphi^\xi(z)-\varphi^\xi(z_0)|}{|z-z_0|} \leq \lim_{z\to z_0}\sup_{t\in(0,1)} 2\sqrt{W^\xi(z_0+t(z-z_0))} = 2\sqrt{W^\xi(z_0)},$$

where in the last step we used the continuity of W^ξ. Thus, this implies

$$|\nabla(\varphi^\xi\circ u_n)(x)| \leq 2\sqrt{W^\xi(u_n(x))}|\nabla u_n(x)| \qquad \text{for a.e. } x\in\Omega. \tag{58}$$

We will now define the constant

$$\sigma^\xi := \varphi^\xi(b) = \inf\left\{\int_{-1}^1 2\sqrt{W^\xi(\gamma(t))}|\gamma'(t)|\,\mathrm{d}t : \gamma \in \mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M), \gamma(-1)=a, \gamma(1)=b\right\}.$$

Since φ^ξ is Lipschitz continuous and $u_n\to u_0$ in L^1, we also have $\varphi^\xi\circ u_n \to \varphi^\xi\circ u_0$ in L^1. We note that

$$\varphi^\xi\circ u_0(x) = \sigma^\xi \mathbb{1}_B(x),$$

since $u_0\in\mathrm{BV}(\Omega;\{a,b\})$. To conclude, using lower semi-continuity of the total variation and (58), we get

$$\begin{aligned}\liminf_{n\to\infty}\int_\Omega 2\sqrt{W^\xi(u_n)}|\nabla u_n|\,\mathrm{d}x &\geq \liminf_{n\to\infty}\int_\Omega |\nabla(\varphi^\xi\circ u_n)|\\ &\geq \int_\Omega|\nabla(\varphi^\xi\circ u_0)| = \sigma^\xi\,\mathrm{Per}(\{u=b\};\Omega).\end{aligned}$$

This proves the claim.

Step 4: Conclusion. Using (54) and (45), we get that

$$\liminf_{n\in\mathbb{N}} F_n^{(1)}(u_n) \geq \sigma^{\xi}\,\mathrm{Per}(\{u=b\};\Omega).$$

Taking the limit as $\xi \to 0$, and using Proposition 41, we get

$$\liminf_{n\in\mathbb{N}} F_n^{(1)}(u_n) \geq \sigma^{\mathrm{h}}\,\mathrm{Per}(\{u=b\};\Omega).$$

This concludes the proof.

□

7. Limsup inequality

The goal of this section is to prove the following proposition.

Proposition 43. *Let $u \in \mathrm{BV}(\Omega;\{a,b\})$. Then, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,2}(\Omega;\mathbb{R}^M)$, such that $u_n \to u$ strongly in $L^1(\Omega;\mathbb{R}^M)$ and*

$$\limsup_{n\to\infty} F_n^{(1)}(u_n) \leq F_\infty^{(1)}(u).$$

In order to prove this proposition, we need some technical results, the first of which is an approximation result for sets (see [6, Lemma 3.1]).

Proposition 44. *Let $E \subset \Omega$ be a set with finite perimeter. Then, there exists a sequence of sets $(E_n)_n$ with $E_n \in \mathbb{R}^N$ such that*

- *$\partial E_n \cap \Omega$ is of class C^2;*
- *$\mathcal{H}^{N-1}(\partial E_n \cap \partial\Omega) = 0$;*
- *$\mathbb{1}_{E_n} \to \mathbb{1}_E$ in $L^1(\Omega)$;*
- *$\lim_{n\to\infty} F_\infty^{(1)}(\mathbb{1}_{E_n}) = F_\infty^{(1)}(\mathbb{1}_E)$.*

Corollary to this proposition, we also need the following result about a geometric property of the tubular neighborhoods of a C^2 surface in $\mathbb{R}^N$.

Lemma 45. *Let $A \subset \mathbb{R}^N$ be an open bounded set with C^2 boundary, and let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary such that $\mathcal{H}^{N-1}(\partial\Omega \cap \partial A) = 0$. Define the function $h\colon \mathbb{R}^N \to \mathbb{R}$ as*

$$h(x) := \begin{cases} -\mathrm{dist}(x,\partial A) & x \in A \\ \mathrm{dist}(x,\partial A) & x \notin A. \end{cases}$$

Then h is Lipschitz continuous and $|\mathrm{D}h(x)| = 1$ for a.e. $x \in \mathbb{R}^N$.
Moreover, define $S_t := \{x \in \mathbb{R}^N : h(x) = t\}$. Then it holds

$$\lim_{t\to 0} \mathcal{H}^{N-1}(S_t \cap \Omega) = \mathcal{H}^{N-1}(\partial A \cap \Omega). \tag{59}$$

The next result is the key point in the original procedure by Modica-Mortola, as it ensures the existence of a reparameterization of the optimal curve up to a small error (see [34, proof of Proposition 2], [18, Lemma 6.3]).

Lemma 46. *Fix $\lambda > 0$, $\varepsilon > 0$. Let $\gamma \in C^1([-1,1];\mathbb{R}^M)$, with $\gamma(-1) = a$, $\gamma(1) = b$, and $\gamma'(s) \neq 0$ for all $s \in (-1,1)$. Then, there exist $\tau > 0$ and $C > 0$ with*

$$C\varepsilon \leq \tau \leq \frac{\varepsilon}{\sqrt{\lambda}} \int_{-1}^{1} |\gamma'(s)|\,\mathrm{d}s,$$

and $g \in C^1((-\tau,\tau);[-1,1])$ such that

$$(g'(t))^2 = \frac{\lambda + W^{\mathrm{h}}(\gamma(g(t)))}{\varepsilon^2 |\gamma'(g(t))|^2} \tag{60}$$

for all $t \in (-\tau, \tau)$, $g(-\tau) = -1$, $g(\tau) = 1$, *and*

$$\int_{-\tau}^{\tau} \left[\frac{1}{\varepsilon} W^h(\gamma(g(t))) + \varepsilon |\gamma'(g(t))|^2 (g'(t))^2 \right] \mathrm{d}s \le \int_{-1}^{1} 2\sqrt{W^{\mathrm{h}}(\gamma(s))} |\gamma'(s)| \, \mathrm{d}s + 2\sqrt{\lambda} \int_{-1}^{1} |\gamma'(s)| \, \mathrm{d}s \tag{61}$$

Let $A := \{x \in \Omega : u(x) = a\}$. Note that since $u \in \mathrm{BV}(\Omega; \{a, b\})$, it follows that A is set of finite perimeter.

Proof. **Step 0:** Using Proposition 44 and a diagonalization argument, without loss of generality we prove the result for $u \in \mathrm{BV}(\Omega; \{a, b\})$ such that $\partial A \cap \Omega$ is of class C^2 and such that

$$\mathcal{H}^{N-1}(\partial A \cap \partial \Omega) = 0.$$

Step 1: Let $u \in \mathrm{BV}(\Omega; \{a, b\})$ as in Step 1. Fix $\varsigma > 0$, and let $\gamma \in C^1([-1, 1]; \mathbb{R}^M)$ with $\gamma(-1) = a$, $\gamma(1) = b$, be such that

$$\int_{-1}^{1} 2\sqrt{W^{\mathrm{h}}(\gamma(s))} |\gamma'(s)| \, \mathrm{d}s \le \sigma_{\mathrm{h}} + \varsigma. \tag{62}$$

Without loss of generality, we can always choose γ such that $\gamma'(s) \neq 0$ for all $s \in (-1, 1)$. Therefore we can apply Lemma 46 with

$$\varepsilon := \varepsilon_n, \qquad \lambda := \left(\frac{\varsigma}{L(\gamma)} \right)^2,$$

where $L(\gamma)$ is the length of the curve γ, given by

$$L(\gamma) := \int_{-1}^{1} |\gamma'(s)| \, \mathrm{d}s < +\infty.$$

Therefore, for each $n \in \mathbb{N}$, we have τ_n and $g_n \in C^1((-\tau_n, \tau_n); [-1, 1])$ such that (60) and (61) hold. Let now $\mathrm{dist}(\cdot, \partial A) \colon \mathbb{R}^N \to \mathbb{R}$ be the signed distance function from ∂A. Since ∂A is of class C^2, by a classical result we know that $\mathrm{dist}(\cdot, \partial A)$ is of class C^1. Therefore,for every $n \in \mathbb{N}$ we define $u_n \colon \Omega \to \mathbb{R}^M$ as

$$u_n(x) := \begin{cases} a & \mathrm{dist}(x, \partial A) < -\tau_n, \\ \gamma(g_n(\mathrm{dist}(x, \partial A))) & |\mathrm{dist}(x, \partial A)| \le \tau_n, \\ b & \mathrm{dist}(x, \partial A) > \tau_n. \end{cases} \tag{63}$$

Observe that $c_1 \varepsilon_n \le \tau_n \le c_2 \varepsilon_n$, therefore $\tau_n \to 0$ as $n \to \infty$. This implies that $u_n \to u$ strongly in $L^1(\Omega; \mathbb{R}^M)$.

Let us now prove the convergence of the energies. We define

$$A_n := \{x \in \Omega : |\mathrm{dist}(x, \partial A)| \le \tau_n\} \implies |A_n| \le c\tau_n. \tag{64}$$

For each $n \in \mathbb{N}$, we partition A_n into four disjoint sets. To do that, we first have to partition the set of generators at scale δ_n into two disjoint subsets:

$$I_n := \{\xi_1 \in G_1 : \delta_n (\xi_1 + Q_1) \subset A_n\},$$
$$I_n^{\mathrm{c}} := \{\xi_1 \in G_1 : \delta_n (\xi_1 + Q_1) \cap \partial A_n \neq \emptyset\}.$$

In this way we divided the periodicity cells at scale δ_n between those intersecting ∂A_n, and those who are strictly included in A_n. For each $\xi_1 \in I_n^{\mathrm{c}}$ we now define a partition of the generators at scale $\frac{\eta_n}{\delta_n}$:

$$J_n(\xi_1) := \left\{ \xi_2 \in G_2 : \delta_n \left(\xi_1 + \frac{\eta_n}{\delta_n} (\xi_2 - \iota_{2,\eta} + Q_2) \right) \subset A_n \right\},$$
$$J_n^{\mathrm{c}}(\xi_1) := \left\{ \xi_2 \in G_2 : \delta_n \left(\xi_1 + \frac{\eta_n}{\delta_n} (\xi_2 - \iota_{2,\eta} + Q_2) \right) \cap \partial A_n \neq \emptyset \right\}.$$

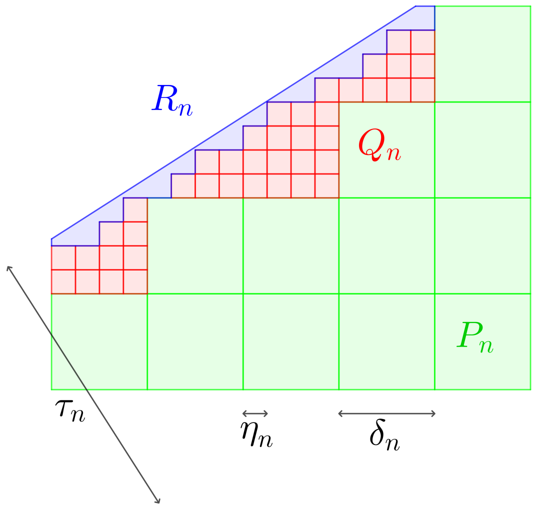

Figure 5. Subdivision of the interface layer

In this way we also divided the periodicity cells at scale $\frac{\eta_n}{\delta_n}$ between those intersecting ∂A_n, and those strictly included in $\delta_n(\xi_1+Q_1)$, for $\xi_1 \in I_n^{\mathrm{c}}$.

With these generators we can partition A_n into three disjoint sets, given by (see Figure 5 for a simplified idea of the subdivision):

$$\begin{aligned} P_n &\coloneqq \bigcup_{\xi_1\in I_n} \delta_n(\xi_1+Q_1),\\ Q_n &\coloneqq \bigcup_{\xi_1\in I_n^{\mathrm{c}}} \bigcup_{\xi_2\in J_n} \delta_n\left(\xi_1+\frac{\eta_n}{\delta_n}\left(\xi_2-\iota_{2,\eta}+Q_2\right)\right),\\ R_n &\coloneqq A_n\setminus(P_n\cup Q_n). \end{aligned}$$

Using (64) we can easily deduce the following estimates.

$$|P_n|\le c\tau_n, \qquad |Q_n|\le c\delta_n, \qquad |R_n|\le c\eta_n. \tag{65}$$

Our objective now is to estimate the value of $F_n^{(1)}(u_n)$, which, using the definition of the recovery sequence, is equal to

$$\begin{aligned} F_n^{(1)}(u_n) &= \int_\Omega \left[\frac{1}{\varepsilon_n}W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n\right)+\varepsilon_n|\nabla u_n|^2\right]\mathrm{d}x\\ &= \int_{A_n} \left[\frac{1}{\varepsilon_n}W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n\right)+\varepsilon_n|\nabla u_n|^2\right]\mathrm{d}x. \end{aligned}$$

By adding and subtracting $\frac{1}{\varepsilon_n}W^{\mathrm{h}}(u_n)$ inside the integral we get

$$\begin{aligned} &\int_{A_n} \left[\frac{1}{\varepsilon_n}W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n\right)+\varepsilon_n|\nabla u_n|^2\right]\mathrm{d}x\\ &\qquad\le \int_{A_n}\left[\frac{1}{\varepsilon_n}W^{\mathrm{h}}(u_n)+\varepsilon_n|\nabla u_n|^2\right]\mathrm{d}x+\frac{1}{\varepsilon_n}\int_{A_n}\left|W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n\right)-W^{\mathrm{h}}(u_n)\right|\mathrm{d}x. \end{aligned}\tag{66}$$

First of all, we want to prove that:

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n}\int_{A_n}\left|W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n\right)-W^{\mathrm{h}}(u_n)\right|\mathrm{d}x=0. \tag{67}$$

Note that, as we wanted, the sets P_n, Q_n and R_n are pairwise disjoint and $A_n = P_n \cup Q_n \cup R_n$. Thus, we get that

$$\begin{aligned}&\int_{A_n} W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n\right) \mathrm{d}x \\ &\qquad = \int_{P_n} W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n\right) \mathrm{d}x + \int_{Q_n} W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n\right) \mathrm{d}x + \int_{R_n} W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n\right) \mathrm{d}x.\end{aligned}$$

We now define

$$z_n := \delta_n \xi_1 + \eta_n \xi_2 - \eta_n \iota_{2,\eta},$$

and also the following sum operator:

$$\sum_{\xi_1,\xi_2} := \sum_{\xi_1 \in I_n} \sum_{\xi_2 \in \Xi_2} + \sum_{\xi_1 \in I_n^c} \sum_{\xi_2 \in J_n(\xi_1)}, \tag{68}$$

which incorporates the integral over P_n and Q_n. Therefore, we get

$$\begin{aligned}&\int_{P_n} W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n\right) \mathrm{d}x + \int_{Q_n} W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n\right) \mathrm{d}x + \int_{R_n} W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n\right) \mathrm{d}x \\ &\quad = \sum_{\xi_1,\xi_2} \int_{\delta_n\xi_1 + \delta_n \widehat{Q}_{1,\delta}} \int_{\frac{\eta_n}{\delta_n}\xi_2 - \frac{\eta_n}{\delta_n}\iota_{2,\eta} + \frac{\eta_n}{\delta_n} Q_2} \int_{Q_2} W(y_1, y_2, \mathcal{U}^2 u_n) \,\mathrm{d}y_2 \,\mathrm{d}y_1 \,\mathrm{d}x + \int_{R_n} W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n\right) \mathrm{d}x\end{aligned}$$

By using the definition of the homogenized $W^{\mathrm{h}}(p)$ given by

$$W^{\mathrm{h}}(z) := \int_{Q_1} \int_{Q_2} W(y_1, y_2, z) \,\mathrm{d}y_2 \,\mathrm{d}y_1,$$

and the definition of z_n given above, we get

$$\begin{aligned}&\int_{A_n} \left| W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n(x)\right) - W^{\mathrm{h}}(u_n(x)) \right| \mathrm{d}x \\ &\quad \leq \sum_{\xi_1,\xi_2} \int_{\delta_n\xi_1 + \delta_n \widehat{Q}_{1,\delta}} \int_{\frac{\eta_n}{\delta_n}\xi_2 - \frac{\eta_n}{\delta_n}\iota_{2,\eta} + \frac{\eta_n}{\delta_n} Q_2} \int_{Q_2} |W(y_1, y_2, u_n(z_n + \eta_n y_2)) - W(y_1, y_2, u_n(z_n))| \,\mathrm{d}y_2 \,\mathrm{d}y_1 \,\mathrm{d}x \\ &\qquad + \sum_{\xi_1,\xi_2} \int_{\delta_n\xi_1 + \delta_n \widehat{Q}_{1,\delta}} \int_{\frac{\eta_n}{\delta_n}\xi_2 - \frac{\eta_n}{\delta_n}\iota_{2,\eta} + \frac{\eta_n}{\delta_n} Q_2} \int_{Q_2} |W(y_1, y_2, u_n(z_n)) - W(y_1, y_2, u_n(x))| \,\mathrm{d}y_2 \,\mathrm{d}y_1 \,\mathrm{d}x \\ &\qquad + \int_{R_n} \left| W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n(x)\right) - W^{\mathrm{h}}(u_n(x)) \right| \mathrm{d}x \\ &\quad =: J_n^1 + J_n^2 + J_n^3.\end{aligned} \tag{69}$$

Before estimating these terms, we need these two further estimates.
Take $x \in \delta_n\xi_1 + \delta_n\widehat{Q}_{1,\eta}$, z_n defined as above as $z_n = \delta_n\xi_1 + \eta_n\xi_2 - \eta_n\iota_{2,\eta}$, and $y_2 \in Q_2$. We need to estimate $|u_n(x) - u_n(z_n)|$ and $|u_n(z_n + \eta_n y_2) - u_n(z_n)|$. From Lemma 46 we know that $|g_n'| \leq \frac{C}{\varepsilon_n}$, therefore, by denoting with $\omega_\gamma \colon [0, +\infty) \to [0, +\infty)$ the modulus of continuity of γ, we can write

$$\begin{aligned}|u_n(x) - u_n(z_n)| &= |\gamma\left(g_n\left(\mathrm{dist}(x, \partial A)\right)\right) - \gamma\left(g_n\left(\mathrm{dist}(z_n, \partial A)\right)\right)| \\ &\leq \omega_\gamma\left(g_n\left(\mathrm{dist}(x, \partial A)\right) - g_n\left(\mathrm{dist}(z_n, \partial A)\right)\right) \\ &\leq \omega_\gamma\left(\frac{1}{\varepsilon_n}\left|\mathrm{dist}(x, \partial A) - \mathrm{dist}(z_n, \partial A)\right|\right) \\ &\leq \omega_\gamma\left(\frac{1}{\varepsilon_n}|x - z_n|\right) \\ &\leq \omega_\gamma\left(C\frac{\delta_n}{\varepsilon_n}\right).\end{aligned}$$

For the other estimate it is analogous, and the only difference is that $|z_n + \eta_n y_2 - z_n| \leq C\eta_n$. Therefore we get

$$\begin{aligned}
|u_n(z_n + \eta_n y_2) - u_n(z_n)| &= |\gamma\left(g_n\left(\operatorname{dist}(z_n + \eta_n y_2, \partial A)\right)\right) - \gamma\left(g_n\left(\operatorname{dist}(z_n, \partial A)\right)\right)| \\
&\leq \omega_\gamma\left(g_n\left(\operatorname{dist}(z_n + \eta_n y_2, \partial A)\right) - g_n\left(\operatorname{dist}(z_n, \partial A)\right)\right) \\
&\leq \omega_\gamma\left(\frac{1}{\varepsilon_n}\left|\operatorname{dist}(z_n + \eta_n y_2, \partial A) - \operatorname{dist}(z_n, \partial A)\right|\right) \\
&\leq \omega_\gamma\left(\frac{1}{\varepsilon_n}\eta_n\left|y_2\right|\right) \\
&\leq \omega_\gamma\left(C\frac{\eta_n}{\varepsilon_n}\right).
\end{aligned}$$

We are now ready to estimate each term on the right-hand side of (69). For J_n^3 we have the simple estimate:

$$\begin{aligned}
|J_n^3| &= \int_{R_n}\left|W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n(x)\right) - W^{\mathrm{h}}\left(u_n(x)\right)\right| \mathrm{d}x \\
&\leq \int_{R_n}\left|W\left(\frac{x}{\delta_n}, \frac{x}{\eta_n}, u_n(x)\right)\right| \mathrm{d}x + \int_{R_n}\left|W^{\mathrm{h}}\left(u_n(x)\right)\right| \mathrm{d}x \\
&\leq 2C_M|R_n| \\
&\leq c\,\eta_n, \qquad (70)
\end{aligned}$$

where the last step follows from (65).

For the estimate of J_n^2, we denote by ω_W the modulus of continuity of W in $Q_1 \times Q_2 \times B_M(0)$. Let H_2^n be the cardinality of $(I_n \times \Xi_2) \cup (I_n^{\mathrm{c}} \times J_n)$. From our costruction it follows that:

$$\lim_{n\to\infty} H_2^n \left\lfloor \frac{|A_n|}{\eta_n^N} \right\rfloor^{-1} = 1.$$

Therefore we get:

$$\begin{aligned}
|J_n^2| &\leq \sum_{\xi_1,\xi_2} \int_{\delta_n\xi_1+\delta_n\widehat{Q}_{1,\delta}} \int_{\frac{\eta_n}{\delta_n}\xi_2-\frac{\eta_n}{\delta_n}\iota_{2,\eta}+\frac{\eta_n}{\delta_n}Q_2} \int_{Q_2} |W(y_1, y_2, u_n(z_n)) - W(y_1, y_2, u_n(x))| \,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x \\
&\leq \sum_{\xi_1,\xi_2} \int_{\delta_n\xi_1+\delta_n\widehat{Q}_{1,\delta}} \int_{\frac{\eta_n}{\delta_n}\xi_2-\frac{\eta_n}{\delta_n}\iota_{2,\eta}+\frac{\eta_n}{\delta_n}Q_2} \int_{Q_2} \omega_W\left(y_1, y_2, |u_n(z_n) - u_n(x)|\right) \mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x \\
&\leq \sum_{\xi_1,\xi_2} \int_{\delta_n\xi_1+\delta_n\widehat{Q}_{1,\delta}} \int_{\frac{\eta_n}{\delta_n}\xi_2-\frac{\eta_n}{\delta_n}\iota_{2,\eta}+\frac{\eta_n}{\delta_n}Q_2} \int_{Q_2} \omega_W\left(y_1, y_2, \omega_\gamma\left(c\frac{\delta_n}{\varepsilon_n}\right)\right) \mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x \\
&\leq H_2^n\eta_n^N \int_{Q_1}\int_{Q_2} \omega_W\left(y_1, y_2, \omega_\gamma\left(c\frac{\delta_n}{\varepsilon_n}\right)\right) \mathrm{d}y_2\,\mathrm{d}y_1 \\
&\leq c\,\varepsilon_n \int_{Q_1}\int_{Q_2} \omega_W\left(y_1, y_2, \omega_\gamma\left(c\frac{\delta_n}{\varepsilon_n}\right)\right) \mathrm{d}y_2\,\mathrm{d}y_1. \qquad (71)
\end{aligned}$$

From the definition of $\omega_W(y_1, y_2, \cdot)$, it follows that for almost every $y_1 \in Q_1$, $y_2 \in Q_2$ it holds

$$\lim_{n\to\infty} |W(y_1, y_2, u_n(z_n)) - W(y_1, y_2, u_n(x))| \leq \omega_W\left(y_1, y_2, \omega_\gamma\left(c\frac{\delta_n}{\varepsilon_n}\right)\right).$$

Moreover, by the definition of u_n with the boundedness assumption on W, we have that for every $n \in \mathbb{N}$, for almost every $y_1 \in Q_1$, $y_2 \in Q_2$, there exists $C > 0$ such that:

$$|W(y_1, y_2, u_n(z_n)) - W(y_1, y_2, u_n(x))| \leq C.$$

Using the Dominated Convergence Theorem, this implies that

$$\lim_{n\to\infty}\int_{Q_1}\int_{Q_2}\omega_W\left(y_1,y_2,\omega_\gamma\left(c\frac{\delta_n}{\varepsilon_n}\right)\right)\,\mathrm{d}y_2\,\mathrm{d}y_1=0. \tag{72}$$

For J^1_n, we have an analogous estimate:

$$\begin{aligned}
|J^1_n| &\le \sum_{\xi_1,\xi_2}\int_{\delta_n\xi_1+\delta_n\widehat{Q}_{1,\delta}}\int_{\frac{\eta_n}{\delta_n}\xi_2-\frac{\eta_n}{\delta_n}\iota_{2,\eta}+\frac{\eta_n}{\delta_n}Q_2}\int_{Q_2}|W(y_1,y_2,u_n(z_n+\eta_n y_2))-W(y_1,y_2,u_n(z_n))|\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x\\
&\le \sum_{\xi_1,\xi_2}\int_{\delta_n\xi_1+\delta_n\widehat{Q}_{1,\delta}}\int_{\frac{\eta_n}{\delta_n}\xi_2-\frac{\eta_n}{\delta_n}\iota_{2,\eta}+\frac{\eta_n}{\delta_n}Q_2}\int_{Q_2}\omega_W\left(y_1,y_2,|u_n(z_n+\eta_n y_2)-u_n(z_n)|\right)\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x\\
&\le \sum_{\xi_1,\xi_2}\int_{\delta_n\xi_1+\delta_n\widehat{Q}_{1,\delta}}\int_{\frac{\eta_n}{\delta_n}\xi_2-\frac{\eta_n}{\delta_n}\iota_{2,\eta}+\frac{\eta_n}{\delta_n}Q_2}\int_{Q_2}\omega_W\left(y_1,y_2,\omega_\gamma\left(c\frac{\eta_n}{\delta_n}\right)\right)\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}x\\
&\le H_2^n\eta_n^N\int_{Q_1}\int_{Q_2}\omega_W\left(y_1,y_2,\omega_\gamma\left(c\frac{\eta_n}{\delta_n}\right)\right)\mathrm{d}y_2\,\mathrm{d}y_1\\
&\le c\,\varepsilon_n\int_{Q_1}\int_{Q_2}\omega_W\left(y_1,y_2,\omega_\gamma\left(c\frac{\eta_n}{\delta_n}\right)\right)\mathrm{d}y_2\,\mathrm{d}y_1.
\end{aligned} \tag{73}$$

Same as before, we also know that

$$\lim_{n\to\infty}\int_{Q_1}\int_{Q_2}\omega_W\left(y_1,y_2,\omega_\gamma\left(c\frac{\eta_n}{\varepsilon_n}\right)\right)\,\mathrm{d}y_2\,\mathrm{d}y_1=0. \tag{74}$$

Thus, from (69), (74), (72) and (70) we obtain (67).

Then, from (66), the coarea formula and (59), we get:

$$\begin{aligned}
&\limsup_{n\to\infty}F^{(1)}_n(u_n)=\limsup_{n\to\infty}\int_\Omega\left[\frac{1}{\varepsilon_n}W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n\right)+\varepsilon_n|\nabla u_n|^2\right]\mathrm{d}x\\
&=\limsup_{n\to\infty}\int_{A_n}\left[\frac{1}{\varepsilon_n}W\left(\frac{x}{\delta_n},\frac{x}{\eta_n},u_n\right)+\varepsilon_n|\nabla u_n|^2\right]\mathrm{d}x\\
&\le\limsup_{n\to\infty}\int_{A_n}\left[\frac{1}{\varepsilon_n}W^{\mathrm{h}}(u_n)+\varepsilon_n|\nabla u_n|^2\right]\mathrm{d}x\\
&=\limsup_{n\to\infty}\int_{-\tau_n}^{\tau_n}\left[\frac{1}{\varepsilon_n}W^{\mathrm{h}}(\gamma(g_n(s)))+\varepsilon_n|\gamma'(g_n(s))|^2|g_n'(s)|^2\right]\mathcal{H}^{N-1}(\{\mathrm{dist}(x,\partial A)=s\})\,\mathrm{d}s\\
&\le\limsup_{n\to\infty}\sup_{|s|\le\tau_n}\mathcal{H}^{N-1}(\{\mathrm{dist}(x,\partial A)=s\})\int_{-\tau_n}^{\tau_n}\left[\frac{1}{\varepsilon_n}W^{\mathrm{h}}(\gamma(g_n(s)))+\varepsilon_n|\gamma'(g_n(s))|^2|g_n'(s)|^2\right]\mathrm{d}s\\
&\le\limsup_{n\to\infty}\sup_{|s|\le\tau_n}\mathcal{H}^{N-1}(\{\mathrm{dist}(x,\partial A)=s\})\left[\int_{-1}^{1}2\sqrt{W^{\mathrm{h}}(\gamma(t))}|\gamma'(t)|\,\mathrm{d}t+2\sqrt{\lambda}L(\gamma)\right]\\
&\le(\sigma_{\mathrm{h}}+3\varsigma)\limsup_{n\to\infty}\sup_{|s|\le\tau_n}\mathcal{H}^{N-1}(\{\mathrm{dist}(x,\partial A)=s\})\\
&=(\sigma_{\mathrm{h}}+3\varsigma)\,\mathcal{H}^{N-1}(\partial A\cap\Omega),
\end{aligned}$$

where in the last inequality we used Lemma 46 and (62). We conclude by arbitrariness of ς. □

8. Mass-constrained case

The goal of this section is to prove Theorem 13. In the case of a mass constrain, the proof needs to change slightly. The procedures for the compactness and liminf inequality remain unchanged, and only the proof for the limsup has to be modified.

Proof. **Step 1:** We now have $u \in \mathrm{BV}(\Omega; \{a, b\})$ with

$$\int_\Omega u \,\mathrm{d}x = ma + (1-m)b.$$

Then, defining as before $A := \{u = a\}$, this implies that $|A| = m$. Since this is a set of finite perimeter, in Proposition 44 we approximated with a sequence $(A_n)_n$ of appropriate C^2 sets. The problem here is that we did not require these sets to satisfy the constraint $|A_n| = m$ for every $n \in \mathbb{N}$. Therefore, this is the first change that needs to be addressed: we thus follow an idea originally due to Ryan Murray.

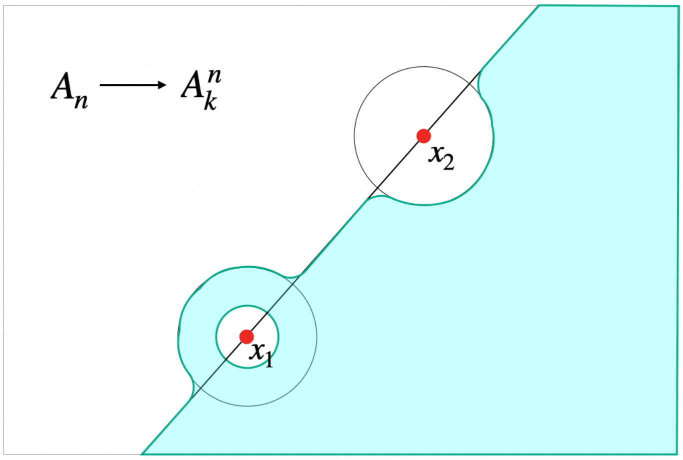

FIGURE 6. Idea of the construction for the C^2 approximations.

Since A is a set of finite perimeter, we can take its reduced boundary $\partial^* A$. Let's now take $x_1, x_2 \in \partial^* A$, and $k \in \mathbb{N}$ and define

$$D_k := \left(A \cup B\left(x_1, \frac{1}{k}\right)\right) \setminus B\left(x_2, \frac{1}{k}\right).$$

it is possible to see that

$$\mathbb{1}_{D_k} \to \mathbb{1}_A \quad \text{in } L^1(\Omega; \mathbb{R}) \quad \text{as } k \to \infty.$$

Moreover, we can check that

$$\begin{aligned}
|\mathrm{D}\mathbb{1}_{D_k}|(\Omega) &= \mathcal{H}^{N-1}(\Omega \cap \partial^* D_k) \\
&\leq \mathcal{H}^{N-1}(\Omega \cap \partial^* A) + \mathcal{H}^{N-1}\left(\Omega \cap B\left(x_1, \frac{1}{k}\right)\right) + \mathcal{H}^{N-1}\left(\Omega \cap B\left(x_2, \frac{1}{k}\right)\right) \\
&\to \mathcal{H}^{N-1}(\Omega \cap \partial^* A) \\
&= |\mathrm{D}\mathbb{1}_A|(\Omega).
\end{aligned}$$

Therefore we also have

$$\lim_{k\to\infty} \mathrm{Per}(D_k; \Omega) = \mathrm{Per}(A; \Omega).$$

Since $x_1, x_2 \in \partial^* A$, they are points of density $\frac{1}{2}$, that is

$$\lim_{r\to 0} \frac{|A \cap B(x_1, r)|}{|B(x_1, r)|} = 0, \qquad \lim_{r\to 0} \frac{|A \cap B(x_2, r)|}{|B(x_2, r)|} = 0.$$

Therefore, this implies that there exists k large enough such that

$$\left|A \cap B\left(x_1, \frac{1}{k}\right)\right| > \frac{1}{4}\left|B\left(x_1, \frac{1}{k}\right)\right|, \qquad \left|A \cap B\left(x_2, \frac{1}{k}\right)\right| < \frac{3}{4}\left|B\left(x_2, \frac{1}{k}\right)\right|.$$

Let now $(D_k^n)_n$ be the sequence of sets obtained by applying Proposition 44 to the set D_k. Since we have convergence in perimeter and L^1, this implies that for a fixed k, there exists $\widetilde{n}_1(k) \in \mathbb{N}$ such that for every $n \geq \widetilde{n}_1(k)$ we have

$$|\mathrm{Per}\,(D_k;\Omega) - \mathrm{Per}\,(D_k^n;\Omega)| \leq \frac{1}{k}, \qquad \int_\Omega \left|\mathbb{1}_{D_k}(x) - \mathbb{1}_{D_k^n}(x)\right| \,\mathrm{d}x \leq \frac{1}{k}.$$

Since these sets D_k^n are obtained by using a standard mollifying procedure and taking a super-level set, we know that there exists $\widetilde{n}_2(k) \in \mathbb{N}$ such that for every $n \geq \max\{\widetilde{n}_1(k), \widetilde{n}_2(k)\}$ we have both the previous inequalities and also

$$B\left(x_1, \left(\frac{4}{5}\right)^{\frac{1}{N}} \frac{1}{k}\right) \subset D_k^n, \qquad B\left(x_2, \left(\frac{4}{5}\right)^{\frac{1}{N}} \frac{1}{k}\right) \subset \Omega \setminus D_k^n.$$

This holds because

$$\left(\frac{4}{5}\right)^{\frac{1}{N}} < 1 \quad \forall N \in \mathbb{N}.$$

We have now three cases to distinguish between. If $|D_k^n| = m$ then we do not have to do anything for now. Assume now that $|D_k^n| > m$. Define $r_k^n > 0$ to be the radius such that

$$|B\,(x_1, r_k^n)| = |D_k^n| - m > 0,$$

and define

$$A_k^n \coloneqq D_k^n \setminus B\,(x_1, r_k^n)\,.$$

We now want to prove that

$$r_k^n < \left(\frac{4}{5}\right)^{\frac{1}{N}} \frac{1}{k}.$$

Since we know that $|A| = m$ and

$$\left|A \cap B\left(x_1, \frac{1}{k}\right)\right| > \frac{1}{4}\left|B\left(x_1, \frac{1}{k}\right)\right|,$$

we have

$$\begin{aligned}
|D_k| &= |A| + \left|B\left(x_1, \frac{1}{k}\right) \setminus A\right| - \left|B\left(x_2, \frac{1}{k}\right) \cap A\right| \\
&\leq |A| + \left|B\left(x_1, \frac{1}{k}\right)\right| - \left|B\left(x_1, \frac{1}{k}\right) \cap A\right| - \left|B\left(x_2, \frac{1}{k}\right) \cap A\right| \\
&\leq |A| + \left|B\left(x_1, \frac{1}{k}\right)\right| - \frac{1}{4}\left|B\left(x_1, \frac{1}{k}\right)\right| \\
&= |A| + \frac{3}{4}\left|B\left(x_1, \frac{1}{k}\right)\right| \\
&= m + \left|B\left(x_1, \left(\frac{3}{4}\right)^{\frac{1}{N}} \frac{1}{k}\right)\right|.
\end{aligned}$$

Therefore, for n large enough we also get

$$|B\,(x_1, r_k^n)| = |D_k^n| - m \leq \left|B\left(x_1, \left(\frac{3}{4}\right)^{\frac{1}{N}} \frac{1}{k}\right)\right| < \left|B\left(x_1, \left(\frac{4}{5}\right)^{\frac{1}{N}} \frac{1}{k}\right)\right|,$$

therefore we got the estimate on r_k^n. This in particular implies that the set A_k^n is also C^2, and it follows that

$$|A_k^n| = |D_k^n| - |B\,(x_1, r_k^n)| = |D_k^n| - |D_k^n| + m = m.$$

Assume now that $|D_k^n| < m$. Define $r_k^n > 0$ to be the radius such that

$$|B\,(x_1, r_k^n)| = m - |D_k^n| > 0,$$

and define

$$A_k^n \coloneqq D_k^n \cup B\left(x_2, r_k^n\right).$$

We now want to prove that

$$r_k^n < \left(\frac{4}{5}\right)^{\frac{1}{N}} \frac{1}{k}.$$

Since we know that $|A| = m$ and

$$\left| A \cap B\left(x_2, \frac{1}{k}\right)\right| < \frac{3}{4}\left| B\left(x_2, \frac{1}{k}\right)\right|,$$

we have

$$\begin{aligned}
|D_k| &= |A| + \left| B\left(x_1, \frac{1}{k}\right) \setminus A\right| - \left| B\left(x_2, \frac{1}{k}\right) \cap A\right| \\
&\geq |A| - \frac{3}{4}\left| B\left(x_2, \frac{1}{k}\right)\right| \\
&= m - \left| B\left(x_2, \left(\frac{3}{4}\right)^{\frac{1}{N}} \frac{1}{k}\right)\right|.
\end{aligned}$$

Therefore, for n large enough we also get

$$\left|B\left(x_2, r_k^n\right)\right| = m - |D_k^n| \leq \left| B\left(x_2, \left(\frac{3}{4}\right)^{\frac{1}{N}} \frac{1}{k}\right)\right| < \left| B\left(x_2, \left(\frac{4}{5}\right)^{\frac{1}{N}} \frac{1}{k}\right)\right|,$$

therefore we got the estimate on r_k^n. This in particular implies that the set A_k^n is also C^2, and it follows that

$$|A_k^n| = |D_k^n| + \left|B\left(x_2, r_k^n\right)\right| = |D_k^n| + m - |D_k^n| = m.$$

Therefore we just proved that we can always for every k we can modify the sequence of sets obtained from Proposition 44 applied to D_k such that every element of the sequence satisfies the mass constraint. Using a diagonal argument, we can therefore obtain the desired conclusion.
Step 2: Let u_n be defined as in (63). In general it is not true that this function satisfies the mass constraint, therefore we need to modify it accordingly.
Let $N \geq 2$ and define

$$m_n \coloneqq \int_\Omega u_n(x)\,\mathrm{d}x.$$

If, for a given $n \in \mathbb{N}$, we have $m_n = m$, then we are done. Let's suppose that $m_n \neq m$. Let us recall the definition of A_n:

$$A_n \coloneqq \{x \in \Omega : |\mathrm{dist}(x, \partial A)| \leq \tau_n\},$$

which is the set where $u_n(x) \notin \{a, b\}$.

Let $x_0 \in \Omega \setminus A_n$ such that $u_n(x_0) = u(x_0) = a$. Let now $(r_n)_{n\in\mathbb{N}}$ be an infinitesimal sequence and define $B_n \coloneqq B(x_0, r_n)$. We can now modify u_n as follows:

$$v_n(x) \coloneqq \begin{cases} u_n(x) & x \in \Omega \setminus B_n, \\ a + c_n(m_n - m)\left(1 - \frac{|x - x_0|}{r_n}\right) & x \in B_n, \end{cases}$$

where $c_n \in \mathbb{R}$ is to be determined by enforcing the mass constraint.

Therefore we must have

$$\begin{aligned}
m &= \int_\Omega v_n(x)\,\mathrm{d}x \\
&= \int_{\Omega\setminus B_n} u_n(x)\,\mathrm{d}x + \int_{B_n} \left[a + c_n(m_n - m)\left(1 - \frac{|x - x_0|}{r_n}\right)\right] \mathrm{d}x \\
&= \int_\Omega u_n(x)\,\mathrm{d}x - \int_{B_n} u_n(x)\,\mathrm{d}x + \int_{B_n} a\,\mathrm{d}x + c_n(m_n - m)\int_{B_n} \left[1 - \frac{|x - x_0|}{r_n}\right] \mathrm{d}x
\end{aligned}$$

$$= m_n + c_n(m_n - m) N\omega_N r_n^N \int_0^1 (1-s)s^{N-1}\,\mathrm{d}s$$
$$= m_n + c_n(m_n - m) r_n^N \frac{\omega_N}{N+1},$$

which implies

$$c_n = -\frac{N+1}{\omega_N}\frac{1}{r_n^N}.$$

Therefore with this choice of c_n, the function v_n satisfies the mass constraint. We need to check that it still converges in L^1, and that is does not change the energy in the limit. Checking the L^1 convergence is easy, since the way we chose the constant c_n implies

$$\left| \int_{B_n} c_n(m_n - m)\left(1 - \frac{|x-x_0|}{r_n}\right)\mathrm{d}x \right| = |m - m_n| \to 0.$$

For the energy convergence, we need to check that

$$\lim_{n\to\infty} F_n^{(1)}(v_n, B_n) = 0.$$

Let us check first the potential energy term:

$$\int_{B_n} \frac{1}{\varepsilon_n} W\left(a + c_n(m_n - m)\left(1 - \frac{|x-x_0|}{r_n}\right)\right)\mathrm{d}x = \frac{r_n^N}{\varepsilon_n}\int_0^1 s^{N-1} W\left(a + c_n(m_n - m)(1-s)\right)\mathrm{d}s.$$

Since $W(z) \leq C$ if $|z| \leq M$ for (H4), we get that this term behaves like

$$\int_{B_n} \frac{1}{\varepsilon_n} W(v_n)\,\mathrm{d}x \leq C\frac{r_n^N}{\varepsilon_n}.$$

For the gradient energy term, we get

$$\int_{B_n} \varepsilon_n |\nabla v_n|^2\,\mathrm{d}x = \int_{B_n} \varepsilon_n \frac{c_n^2|m_n - m|^2}{r_n^2}\,\mathrm{d}x = C\frac{|m_n - m|^2\varepsilon_n}{r_n^{2+N}}.$$

We have

$$|m_n - m| \leq \int_\Omega |u_n - u|\,\mathrm{d}x = \int_{A_n} |u_n - u|\,\mathrm{d}x \leq |b - a|\mathcal{L}^N(A_n) \leq C\varepsilon_n,$$

therefore we obtain

$$\int_{B_n} \varepsilon_n |\nabla v_n|^2\,\mathrm{d}x \leq C\frac{\varepsilon_n^3}{r_n^{2+N}}.$$

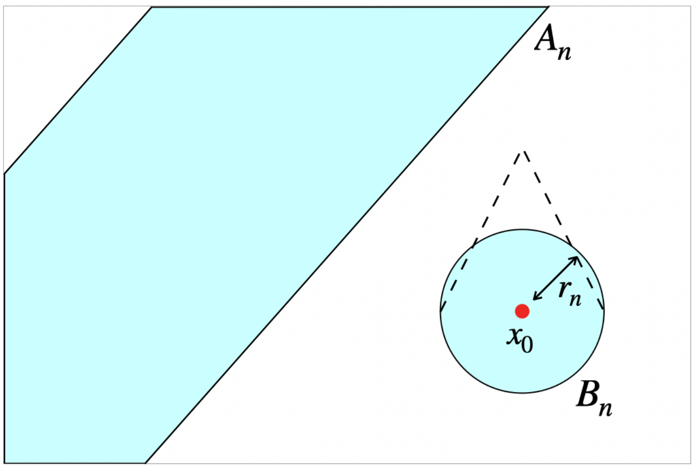

FIGURE 7. Idea of the construction for the recovery sequence.

Summing up we get

$$F_n^{(1)}(v_n, B_n) \le c\frac{r_n^N}{\varepsilon_n} + c\frac{\varepsilon_n^3}{r_n^{2+N}}.$$

Since $\frac{3}{2+N} > \frac{1}{N}$ for $N \ge 2$, it is enough to take $r_n = \varepsilon_n^\alpha$, with

$$\frac{1}{N} < \alpha < \frac{3}{2+N}$$

to prove that $F_n^{(1)}(v_n, B_n) \to 0$, and we conclude.

□

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